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MARKET ANTICIPATIONS, RATIONAL EXPECTATIONS, AND BAYESIAN ANALYSIS*

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1. INTRODUCTION

Muth [1961], Lucas [1975a], and Lucas and Prescott [1971] have stressed that the notion of rational expectations is an *equilibrium* concept and that rational expectations may be regarded as the outcome of some unspecified process of learning and adapting on the part of economic agents. Hence attempts to make such processes explicit may be viewed as analyses of the stability of rational expectations equilibria. In particular one may pose the following stability question: what tendency is there in a model for convergence to a self-fulfilling equilibrium if initially agents have imperfect information concerning the environment in which they must make decisions but use Bayes' rule in accordance with past observations to infer the values of unknown parameters? Several authors including Arrow and Green [1973], Crawford [1973], Cyert and DeGroot [1970], Grossman [1975], and Kihlstrom and Mirman [1975] have addressed this or related questions.

To answer the question in the context of a particular model one must specify precisely what is known by agents at the outset. In this regard two approaches can be distinguished. In the first approach each agent is assumed to have in mind a relatively simple model which expresses the relationships among the variables of concern to him. In this formulation there is imperfect information in that some parameters of the simple model are regarded as unknowns, and learning in that each agent revises over time his estimates of those parameters on the basis of past observations; but the simple model may *not* be consistent, at least at the outset, with the reduced from equations of the actual model. However, these equations may change over time with the actions of agents, and eventually there *may* be convergence to a situation in which the parameters of the simple model are identified and consistent with the actual model. To the extent that there is convergence, such a result is comforting to rational expectation theorists. But it may be argued that the lack of convergence need only indicate that agents were doomed at the outset by the modeler who imposed on them an incorrect view of the world and an apparently limited statistical procedure.[2]

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[2] The statistical procedure is limited in that there is no hypothesis testing of the *model* being used; in particular the actual model is not provided as an alternative. An example of this approach is given in Section 2.

In the second approach, the one adopted here, there is an attempt to move away from an ad hoc specification of the initial models used by agents by postulating that agents know at the outset the reduced form equations of the actual model, though they may be uncertain as to the values of some unknown parameters. This raises the following problem. To correctly estimate the relevant parameters, agents need to have knowledge of the way in which the reduced form equations evolve over time. But the time path itself may depend on the way in which all agents are solving the inference problem. This problem is particularly acute if agents do not share common beliefs, for then parameters representing *market* beliefs become state variables of the model.[3] Of course the problem is made more difficult if the parameters representing market beliefs are also unknowns, for then there is an infinite regress problem in which decisions depend on what agents believe the market believes, on what agents believe the market believes the market believes, and so on. It turns out that both these problems can be resolved in the space of agents' strategies with a Nash equilibrium concept.

In his introductory remarks Muth only asserts that under rational expectations the *average* of subjective distributions of agents of the model equals the objective distribution of the model. He also observes that expectations data reveal considerable cross-sectional differences of opinion. To this last remarks may be added the observation made by Keynes [1965] and others that in actual market situations agents do seem to take into consideration the beliefs of others. In particular Keynes draws an analogy between those who purchase shares on stock markets and those in a contest who select from among many photographs the one which they believe others will select. Of course implicit in Keynes's remarks is the view that this is unfortunate. The activity of forecasting the choice of others is termed *speculation* while the activity of forecasting the prospective yield of assets over their whole life is termed *enterprise*. Keynes also wrote that "Americans are apt to be unduly interested in discovering what average opinion believes average opinion to be".[4] In contrast it is argued in this paper that a concern with the beliefs of others is consistent with rationality though economic considerations may dominate in the limit. In effect what is described is a model with maximizing agents in which the psychology of the market plays an explicit role.

The paper proceeds as follows. The general structure of the models is described in Section 2. A linear partial equilibrium framework is adopted in which risk neutral profit maximizing firms face an exogenous stochastic demand. In this context a self-fulfilling equilibrium is defined. The problem confronting a firm is then formulated in the space of agents' strategies, and a Nash equilibrium is

[3] For a simple resolution of this problem where there is one representative firm (so that implicitly agents share common beliefs) see Cyert and DeGroot [1974]. Lucas [1975b] notes the difficulties which can arise when information sets differ among agents but successfully circumvents the problem by postulating a certain pooling arrangement.

[4] Keynes also argued that third order expectations schemes in which agents act on the basis of beliefs of what others believe others believe were also an important determinant of behavior.

defined. The latter allows for diverse initial prior distributions and learning. Section 3 presents a simple application of the equilibrium concept and establishes convergence to a self-fulfilling equilibrium. Section 4 generalizes the example and resolves the infinite regress problem of beliefs. Section 5 discusses the relationship of a Nash equilibrium in the strategy space to the general notion of a rational expectations equilibrium in these models with Bayesian learning. In Section 6 an attempt is made to relax the specification that agents know initially the reduced form equations of the model up to some unknown parameters without imposing an inconsistent model. Firms are assumed to be completely ignorant of the way other firms *form* expectations but receive noisy and biased signals of market beliefs. The equilibrium of this model converges over time to a self-fulfilling equilibrium in the more general sense that Muth had in mind. Section 7 offers some concluding remarks on the stability of rational expectations equilibria and theories of market psychology with rational agents.

2. STRUCTURE OF THE MODELS AND EQUILIBRIUM CONCEPTS

Following Muth a linear partial equilibrium framework is adopted. Each of a set of firms maximizes expected profit, a function which is quadratic in output. This produced commodity is sold in a competitive market with stochastic demand. Production decisions must be made prior to the realization of the demand. Subsequent to the realization of demand, the market clearing price is the one which would be determined by a Walrasian auctioneer. There are no contingent commodity markets. Under these assumptions the production decision of each firm is a linear function of the expected price. In terms of generality, neither the linearity, nor the partial equilibrium approach, nor the exogenous restriction of markets, nor the risk neutrality is satisfactory. Yet it is hoped that the results of this paper could be obtained in a more general framework.

The set of firms, I, is assumed to be the closed unit interval, $[0, 1]$. Let q_{it} denote the output of firm i at time t. Let P_t denote the market price of this commodity at time t. Output q_{it} must be chosen prior to the realization of P_t. Prior to its realization, each firm i believes that P_t is a real-valued random variable with mean $E_{it}(P_t)$. Note that $E_{it}(\cdot)$ is indexed by i so that the subjective expectations may differ among firms. Each firm i acts to maximize $E_{it}(P_t)q_{it}-(1/2a)(q_{it})^2$ $(a>0)$ with respect to q_{it}. This yields the linear decision rule $\tilde{q}_{it}=aE_{it}(P_t)$. It is assumed that $E_{it}(P_t)$ is a Lebesgue measurable function of $i\in[0, 1]$ so that $\int_0^1 E_{jt}(P_t)dj$ exists and represents what may be termed the market anticipated price at time t. Let S_t denote the quantity of output supplied at time t. Then $S_t = a\int_0^1 E_{jt}(P_t)dj$. As each firm is of Lebesgue measure zero, each regards the market anticipated price and market output as invariant with respect to its own beliefs and actions.[5]

Let D_t denote the demand for the produced good at time t. It is supposed that $D_t = -bP_t + \theta + \varepsilon_t$ where $b > 0$, and $\{\varepsilon_t, t = 1, 2, \ldots\}$ is a sequence of independent and identically distributed random variables, each of which is normally distributed with zero mean and variance σ_ε^2. Parameter θ may or may not be known. Let $P_t(\varepsilon_t)$ denote the market clearing price at time t as a function of ε_t so that (with negative prices permitted),

$$P_t(\varepsilon_t) = \frac{\theta + \varepsilon_t - a\int E_{jt}(P_t)dj}{b}. \tag{1}$$

In a self-fulfilling equilibrium[6]

$$E_{jt}(P_t) = E[P_t(\varepsilon_t)] \qquad \text{for each} \quad j \in I. \tag{i}$$

The expectation on the right is taken with respect to the distribution of ε_t so condition (i) requires that the common expectation of price equal the mean of the statistical distribution which is generated by the model. It follows that in a self-fulfilling equilibrium $E_{jt}(P_t) = \theta/(a+b)$, $j \in I$, $P_t(\varepsilon_t) = [\theta/(a+b)] + \varepsilon_t/b$, and $S_t = a\theta/(a+b)$. As noted in Section 1, Muth has in mind the somewhat weaker condition

$$\int E_{jt}(P_t)dj = \theta/(a+b). \tag{ii}$$

The problem confronting a firm in this framework is now formulated in the space of agents' strategies, and a Nash equilibrium is defined. This approach will allow for diverse initial beliefs, for learning, and for an analysis of the stability question posed earlier. In all of the subsequent models of this paper each firm is assumed to know the model up to some unknown parameters and to have beliefs concerning such parameters. These initial beliefs of firm i are represented by a normal probability distribution which is characterized by a vector of parameters $W_{i1} \in \Omega$. Each firm observes at each time t a vector of variables y_t. Then a decision rule $d_{it}(\cdot)$ for firm i at time t is a function which maps W_{i1} and the history of observables $\{y_1, y_2, \ldots, y_{t-1}\}$ to an output choice q_{it}. That is $\tilde{q}_{it} = d_{it}(W_{i1}; y_1, y_2, \ldots, y_{t-1})$. A set of decision rules $\{d_{it}^*(\cdot), i \in I\}$ is said to constitute a *Nash equilibrium* at time t if for each firm i, $\tilde{q}_{it} = d_{it}^*(W_{i1}; y_1, \ldots, y_{t-1})$ is a maximizing choice of q_{it} given the $d_{jt}^*(\cdot)$, $j \neq i$. Note that a Nash equilibrium at time t requires that each firm know the decision rules $d_{jt}^*(\cdot)$, $j \neq i$, but not neces-

[5] Subsequently $\int x_i d_i$ will denote Lebesgue integration of the function x_i on [0,1]. It will be assumed that the model is specified in such a way that x_i is indeed a Lebesgue integrable function. Also any reference in the text to *all* firms should be understood to mean all $i \in [0,1]$ except for a set of measure zero.

[6] The term self-fulfilling equilibrium is used here in contradistinction to the more general notion of a rational expectations equilibrium as defined for example in Lucas [1975b]. See also the discussion in Section 5 below.

sarily the output choices $\tilde{q}_{jt}, j \neq i$. It should also be noted that the $d^*_{it}(\cdot)$, $i \in I$ with $D_t = -bP_t + \theta + \varepsilon_t$ and $S_t = D_t$ completely determine the motion of the system over time.

The objective in what follows is to find a specification for the $\{d_{it}(\cdot), i \in I\}$ which constitutes a Nash equilibrium in each period. Clearly the nature of such rules will depend on the information structure of the model. In this regard it should be noted that if the parameters a, b, and θ and the distribution of ε_t are known, then the supply decisions of the self-fulfilling equilibrium are Nash equilibrium strategies. Suppose that $\tilde{q}_{jt} = a\theta/(a+b)$ for each $j \neq i$. Then $\int \tilde{q}_{jt} dj = a\theta/(a+b)$ so that $bP_t = \theta + \varepsilon_t - [a\theta/(a+b)]$. Consequently $E_{it}(P_t) = \theta/(a+b)$ so that $\tilde{q}_{it} = a\theta/(a+b)$.

It should also be emphasized that by positing that the model is known by firms up to some unknown parameters and that there exists a Nash equilibrium in each period, the class of models which can be used to analyze learning and convergence is restricted. For example, consider the following model analyzed by Cyert and DeGroot [1974]. Each firm believes that

$$P_t = \rho P_{t-1} + v_t \tag{2}$$

where the $\{v_t;\ t = 1, 2, \ldots\}$ is a sequence of independent but identically distributed normal variates with known zero mean and (known) variance $1/r = \sigma_\varepsilon^2/b^2$. Each firm regards ρ as a fixed but unknown parameter. If the beliefs of the representative firm concerning ρ at the beginning of period t are represented by a normal distribution with mean m_t and variance $1/h_t$, then having observed P_t, the representative firm will have beliefs on ρ at the beginning of period $t+1$ represented by a normal distribution with mean m_{t+1} and variance $1/h_{t+1}$ where

$$m_{t+1} = \frac{h_t m_t + rP_{t-1}P_t}{h_t + rP_{t-1}^2},\ h_{t+1} = h_t + rP_{t-1}^2. \tag{3}$$

It follows from (2) and (3) that for the representative firm

$$E_t(P_t) = m_t P_{t-1}. \tag{4}$$

But assuming that for the model of Section 2, $\theta = 0$ and $v_t = \varepsilon_t/b$, Equation (1) yields the actual reduced form equation

$$P_t = \{[-aE_t(P_t)]/b\} + v_t. \tag{5}$$

Thus, substituting (4) into (5) and using the relationships of (3) with h_1, m_1, and P_0 specified exogenously one obtains

$$P_t = \{[-a\varphi_t(\sigma_\varepsilon^2;\ h_1;\ m_1;\ P_0, P_1, \ldots, P_{t-1})P_{t-1}]/b\} + v_t. \tag{6}$$

Equation (6) is inconsistent with the view that ρ is a fixed parameter. Moreover, suppose a single firm were to become enlightened as to the actual mechanism generating prices, i.e., Equation (6), whereas other firms continue to use Equation (2). Then if $E_t(\rho) \neq 0$ and $P_{t-1} \neq 0$, the choice of output of the enlightened firm would differ from the output choice of all other firms. In short, if all firms were enlightened, the decision rules which map h_1, m_1, and $\{P_0, P_1,\ldots, P_{t-1}\}$ into output choices in accordance with (2), (3), and (4) would not constitute in general Nash equilibrium strategies.[7]

3. CONVERGENCE WITH NASH EQUILIBRIUM DECISION RULES

It is assumed for this section that the parameters a and b and the distribution of ε_t are known by all firms but that the parameter θ is unknown. Each firm i has an initial prior distribution on θ which is normal with mean $m_1(i)$ and variance σ_1^2. Parameter θ is regarded by each firm as independent of ε_t. Let $W_{i1} = [m_1(1), \sigma_1^2]$. It is further assumed that the $\{W_{i1}, i \in I\}$ are known by all firms. P_t is observed each period; S_t and ε_t are unobserved.

The search for Nash equilibrium decision rules for this model is facilitated by the following observation. For firm i the state of the model subsequent to the realization of P_t is completely specified by its own mean beliefs about θ, $m_t(i)$, and (if known) by the mean beliefs of others, $\int m_t(j)dj$. $\tilde{q}_{it}$ should increase with $m_t(i)$; ceteris paribus a high expected demand is equivalent to a high expected price. In contrast, $\tilde{q}_{it}$ should decrease with $\int m_t(j)dj$; ceteris paribus, market supply should increase with optimism about demand, lowering the price expected by firm i. This leads to

PROPOSITION 1. *Suppose that the $\{m_t(i), i \in I\}$ are functions of $\{W_{i1}; P_1, P_2,\ldots, P_{t-1}\}$ and that $\int m_t(j)dj$ is known at time t. Then there exist Nash equilibrium decision rules at time t of the form $d_{it}(W_{i1}; P_1, P_2,\ldots, P_{t-1}) = \alpha_0 m_t(i) + \alpha_1 \int m_t(j)dj$ with constants $\alpha_0 > 0$, $\alpha_1 < 0$.*

PROOF. Suppose that

$$\tilde{q}_{jt} = \alpha_0 m_t(j) + \alpha_1 \int m_t(k)dk. \tag{7}$$

Then

$$\int \tilde{q}_{jt} dj = (\alpha_0 + \alpha_1) \int m_t(k)dk.$$

[7] If $m_t \to 0$ as $t \to \infty$, then there is convergence to a self-fulfilling equilibrium. Cyert and DeGroot [1974] present the results of several Monte Carlo runs of this model for various specifications of a/b, m_1 and P_0 (holding $r=1$ and $h_0=0$). Though the model apparently converges under some specifications, it is also reported for $a/b=1.5$, $P_0=m_1=1$ the model diverges for five of ten runs in that P_t begins oscillating wildly, increasing in absolute value, and for $a/b=1.75$, $m_1=1$ and $P_0=10$ the model explodes for all ten runs. It should also be noted that with a diffuse prior ($h_1=0$) the estimates $E_t(\rho)$ are those of ordinary least squares. Thus the failure of the model to converge can not be attributed to Bayesian analysis.

Firm i knows that $D_t=\theta-bP_t+\varepsilon_t$ and the market clearing condition $S_t=D_t$. Hence

$$bE_{it}(P_t) = m_t(i) - (\alpha_0 + \alpha_1)\int m_t(j)dj.$$

Therefore

$$\tilde{q}_{it} = \frac{a}{b}\left[m_t(i) - (\alpha_0 + \alpha_1)\int m_t(j)dj\right]. \tag{8}$$

Equations (7) and (8) are consistent with $\alpha_0=a/b$ and $\alpha_1=-a^2/(b)(a+b)$. Hence

$$\tilde{q}_{it} = \frac{a}{b}\left[m_t(i) - \frac{a}{a+b}\int m_t(j)dj\right]. \tag{9}$$

Q.E.D.

It remains to show that the hypotheses of Proposition 1 are satisfied. At time one $\int m_1(j)dj$ is known. Then q_{i1} is chosen in accordance with (9). The supply decisions of other firms are unobserved, but firm i believes that all such decisions are made in accordance with (9); it is on the basis of this belief that the choice of q_{i1} is rationalized. Firm i then observes the market clearing price of period one, P_1, and regards bP_1+S_1 as a realization of $\theta+\varepsilon_1$. Then a posterior normal distribution on θ is formed by firm i with parameters $W_{i2}=[m_2(i), \sigma_2^2]$ where

$$m_2(i) = \frac{\sigma_\varepsilon^2 m_1(i) + \sigma_1^2(bP_1 + S_1)}{\sigma_1^2 + \sigma_\varepsilon^2}, \quad \frac{1}{\sigma_2^2} = \frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_1^2}.$$

In a similar manner firm i can compute $\{W_{j2}, j\in I\}$. The entire process is then repeated.

Under these assumptions $m_t(i)\to\theta$ as $t\to\infty$ for each $i\in I$ with probability one. Hence $S_t\to a\theta/(a+b)$, $P_t\to[\theta/(a+b)]+\varepsilon_t/b$ and $E_{it}(P_t)\to\theta/(a+b)$, $i\in I$. The limiting equilibrium is the self-fulfilling equilibrium.

4. ON THE EXISTENCE OF NASH EQUILIBRIUM DECISION RULES AND THE PROBLEM OF INFINITE REGRESS

This section analyzes a generalized version of the model of Section 3 which allows for less initial information on the beliefs of others. Conditions sufficient to ensure the existence of Nash equilibrium decision rules are presented, and, in the process, a problem of infinite regress is resolved; each firm is shown to make decisions based on what it believes, what it believes the market believes, what it believes the market believes the market believes, and so on. The motion of the system over time is also analyzed.

For the model of this section firms are again assumed to know the parameters a and b and the distribution of ε_t. As before, P_t is observed at time t, and ε_t

and S_t are unobserved. Also as before, each firm has an initial prior on the unknown parameter θ. In contrast, the means of the prior beliefs of others may be unknown. But each firm is assumed to have a well specified distribution on the mean beliefs of others. To describe these beliefs the following notation and terminology is useful. Let $\theta_{0t}=\theta$ for all t. Let the prior of firm i on θ at time t be termed its zero order belief on θ with expectation $m_{0t}(i)$. Let $\theta_{1t}=\int m_{0t}(j)dj$; θ_{1t} is regarded as another parameter of the model. Let the prior of firm i on θ_{1t} at time t be termed its first order belief on θ with expectation $m_{1t}(i)$. Let $\theta_{2t}=\int m_{1t}(j)dj$. θ_{2t} is another parameter of the model. Let the prior of firm i on θ_{2t} be termed its second order belief on θ with expectation $m_{2t}(i)$. Continuing in this manner let $\theta_{nt}=\int m_{n-1,t}(j)dj$ for all integers $n\geq 1$. Let $\theta_t=\{\theta_{nt};\ n\geq 0\}$. Let $m_t(i)=\{m_{nt}(i);\ n\geq 0\}$. Let $\Sigma(\theta_t)$ denote the covariance matrix of θ_t for firm i with $(k+1)$-th row $(n+1)$-th column element $\sigma_{kn}(\theta_t)$.

It is assumed that firm i's prior on θ_1 is distributed normally with mean vector $m_1(i)$ and covariance matrix $\Sigma(\theta_1)$. The latter is held in common by all firms, and this is known by all firms. Parameter θ_1 is regarded as independent of ε_1 for all i. It should be noted that if θ_{n1} is known then θ_{m1} is known by all $m\geq n$. In such cases the prior of each firm i on θ_1 will have finite dimension. In general, for this model, $W_{i1}=[m_1(i),\ \Sigma(\theta_1)]$.

Following the results of Section 3 it is natural to postulate that a consistent decision rule for firm i at time t will be a linear function of its zero, first, second, and possibly higher order mean beliefs on θ. That is, the choice of output of firm i will depend on its mean belief of θ, its mean belief of the mean beliefs of others, its mean belief of the mean beliefs of the mean beliefs of others, and so on. This structure may appear to lead to the problem of infinite regress, but under specified assumption the infinite series converges.[8] In particular consider

PROPOSITION 2. *Suppose that the* $\{m_{nt}(i);\ n\geq 0\}$ *are functions of* $\{W_{i1};\ P_1, P_2,\dots, P_{t-1}\}$. *Suppose also that as* $N\to\infty$, $\sum_{i=0}^{N}\sum_{j=0}^{N}\alpha_i\alpha_j\sigma_{ij}(\theta_t)$ *converges and* $\sum_{n=0}^{N}\alpha_n m_{nt}(j)$ *converges uniformly where* $\alpha_n=(a/b)^{n+1}(-1)^n$. *Then there exist Nash equilibrium decision rules at time t of the form* $\tilde{q}_{it}=\sum_{n=0}^{\infty}\alpha_n m_{nt}(i)$.

PROOF. Suppose that

$$\tilde{q}_{jt} = \sum_{n=0}^{\infty}\alpha_n m_{nt}(j) \tag{10}$$

where $\alpha_n=(a/b)^{n+1}(-1)^n$. Then $\int\tilde{q}_{jt}dj=\sum_{n=0}^{\infty}\alpha_n\theta_{n+1,t}$. (Note that the summation and integration operators can be interchanged.) For each firm i, the sequence $\sum_{n=0}^{N}\alpha_n\theta_{n+1,t}$ converges in distribution to a normal random variable

[8] This approach to the problem infinite regress is also adopted in part by Cyert and DeGroot [1970].

with mean $\sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)$ and variance $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\sigma_{(i+1)(j+1)}(\theta_t)$. Hence $E_{it}\int\tilde{q}_{jt}dj=\sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)$. By market clearing $bE_{it}(P_t)=m_{0t}(i)-\sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)$ so that

$$\tilde{q}_{it} = \frac{a}{b}[m_{0t}(i) - \sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)]\,. \tag{11}$$

Equations (10) and (11) are the same form. Q.E.D.

It is interesting to note that the second hypothesis of Proposition 2 is satisfied if $a<b$ and if the $\{m_{nt}(i);\ i\in I,\ n\geq 0\}$ and the $\{\sigma_{kn}(\theta_t);\ k\geq 0,\ n\geq 0\}$ are uniformly bounded. The standard Cobweb model is also stable if the slope of the supply curve (of the function $q_{i,t+1}=aP_t$) is less in absolute value than the slope of the inverse demand curve. If $a\geq b$, alternative conditions may ensure the convergence of the relevant infinite series. The following result can also be established.

PROPOSITION 3. *Suppose that the* $\{m_{nt}(i);\ n\geq 0\}$ *are functions of* $\{W_{i1};\ P_1, P_2,\ldots, P_{t-1}\}$. *Then there exist Nash equilibrium decision rules (of dimension k) of the form* $\tilde{q}_{it}=\sum_{n=0}^{k}\alpha_n m_{nt}(i)$ *if and only if* $m_{kt}(i)$ *is a known constant over* $i\in I$.

PROOF. The proof follows an obvious modification of the proofs of Propositions 1 and 2. Q.E.D.

Proposition 3 may be viewed as a generalization of Proposition 1; if θ_{1t} is known, then $m_{1t}(i)=\theta_{1t}$ for all $i\in I$.

It remains to analyze the motion of the system over time and to establish in particular that $m_{nt}(i)$ is a well defined function of $(W_{i1};\ P_1, P_2,\ldots, P_{t-1})$ for all i, n, and t.[9] One's ability to say anything in this regard depends on the finding of a solution to an apparently nontrivial fixed point problem, for having observed P_t, firm j must convert his posterior beliefs on θ_t, what the market believed last period, to prior beliefs on θ_{t+1}, what the market believes now. Yet the way the parameters move over time will be determined by the way in which other firms are solving the same problem. Fortunately the recursive way in which θ_t is defined enables one to find a solution to this fixed point problem in a recursive manner.

In general, given the existence of Nash equilibrium decision rules at time t, $bP_t=\theta_{0t}+\varepsilon_t-\sum_{n=0}^{\infty}\alpha_n\theta_{n+1,t}$ where $\alpha_n=(a/b)^{n+1}(-1)^n$. Then, subsequent to the realization of P_t, firm i will have a posterior normal distribution on θ_t with means

$$m_{0,t+1}(i) = m_{0t}(i) + C_0[bP_t - m_{0t}(i) + \sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)] \tag{12}$$

$$\alpha_{k-1}E_{i,t+1}[\theta_{kt}] = \alpha_{k-1}m_{kt}(i) + C_k[bP_t - m_{0t}(i) + \sum_{n=0}^{\infty}\alpha_n m_{n+1,t}(i)],\ k\geq 1 \tag{13}$$

[9] Proposition 3 may suggest the possibility that if $m_{kt}(i)$ is a known constant over $i\in I$, then $m_{k\tau}(i)$ will be a known constant for all $\tau>t$. Unfortunately this need not be so.

where

$$C_k = \frac{\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^n \sigma_{kn}(\theta_t)}{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^n \sigma_{kn}(\theta_t) + \sigma_\varepsilon^2}, \qquad k \geq 0$$

and with covariance matrix of terms

$$\left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^n \sigma^*_{kn}(\theta_t) = \left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^n \sigma_{kn}(\theta_t) - \frac{\left[\sum_{j=0}^{\infty}\left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^j \sigma_{kj}(\theta_t)\right]\left[\sum_{j=0}^{\infty}\left(\frac{a}{b}\right)^j\left(\frac{a}{b}\right)^n \sigma_{jn}(\theta_t)\right]}{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right)^n \sigma_{kn}(\theta_t) + \sigma_\varepsilon^2}.$$

Integration over (12) yields

$$\theta_{1,t+1} = \gamma_{10}P_t + \sum_{n=1}^{\infty}\gamma_{1n}\theta_{nt} \tag{14}$$

for suitably chosen constants $\{\gamma_{1n};\ n \geq 0\}$ in terms of the parameters a, b, σ_ε^2 and $\sigma_{kn}(\theta_t)$. Taking the expectation for firm i in (14) with respect to the posterior distribution of θ_t and substituting from (13) yields

$$m_{1,t+1}(i) = \gamma_{20}P_t + \sum_{n=0}^{\infty}\gamma_{2,n+1}m_{nt}(i). \tag{15}$$

Integration over (15) yields

$$\theta_{2,t+1} = \gamma_{20}P_t + \sum_{n=1}^{\infty}\gamma_{2n}\theta_{nt} \tag{16}$$

which is of the same form as (14). Proceeding recursively in this manner,

$$\theta_{k,t+1} = \gamma_{k0}P_t + \sum_{n=1}^{\infty}\gamma_{kn}\theta_{nt}, \qquad k \geq 1. \tag{17}$$

Hence the prior distribution of θ_{t+1} at time $t+1$ is well defined with means and covariance matrix determined from (13), (17), and $\sigma^*_{kn}(\theta_t)$. It follows that under appropriate regularity conditions, $m_{t+1}(i)$ is a well defined function of $\{W_{i1};\ P_1, P_2, \ldots, P_t\}$.[10]

It is conjectured that $m_{nt}(i) \to \theta$ as $t \to \infty$ with probability one for all $i \in I$ and $n \geq 0$. Such a conjecture is easily proved in a model in which S_t is also observed each period. For then the equation

$$bP_t + S_t = \theta + \varepsilon_t \tag{18}$$

[10] Certain regularity conditions are needed to ensure the existence of Nash equilibria in each period. It has also been assumed implicitly in the text that all terms involved in the Bayesian updating procedure are well defined.

can be used to update priors about θ, and the equation

$$S_t = \sum_{n=0}^{\infty} \alpha_n \theta_{n+1,t} \tag{19}$$

can be used in conjunction with (18) in a now familiar recursive manner to form a prior distribution on the $\theta_{n,t+1}$. It is clear from (18) that $m_{0t}(i) \to \theta$ as $t \to \infty$ with probability one regardless of θ_1. Consequently $m_{nt}(i) \to \theta$ as $t \to \infty$.

5. ON THE RELATIONSHIP BETWEEN NASH AND RATIONAL EXPECTATIONS EQUILIBRIA

The notion of a Nash equilibrium in the strategy space may be a natural way to extend the notion of a self-fulfilling equilibrium to situations in which agents have different beliefs. But such an extension is not unique. For suppose in the model of Section 4, for example, that all firms act as if

$$(20)\qquad \begin{aligned} bP_t &= \theta + \varepsilon_t - \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^{n+1} (-1)^n \theta_{n+1,t}, && t = 1\\ bP_t &= \theta + \varepsilon_t - \sum_{j=1}^{t-1} \beta_{jt} P_{t-j} - \sum_{n=0}^{\infty} \delta_{nt} \theta_{n+1,1}, && t = 2, 3, \ldots. \end{aligned}$$

By virtue of the existence of a Nash equilibrium there exist values for the β_{jt} and δ_{nt} in terms of the parameters a, b, σ_ε^2, and $\sigma_{kn}(\theta_1)$ such that Equation (20) is self-fulfilling in each period. That is, expectations will be correct if at time t, before the choice of q_{jt}, each firm j takes the expectation in (20) with respect to the distribution of ε_t and, if $t=1$, the prior distribution of θ_1, or if $t>1$, the posterior distribution of θ_1 conditional on $\{P_1, P_2, \ldots, P_{t-1}\}$ using Equation (20) of the previous periods. Each firm need not be concerned with the strategies followed by others if all correctly perceive the way prices move over time. This is in keeping with the notion of a rational expectations equilibrium as described for example in Lucas and Prescott [1971]. (It should be noted in passing that in models with Bayesian learning a state of the economy is a specification of the history of all observables.)

This paper has focused on the notion of a Nash equilibrium in the strategy space because the consequent analysis makes it clear that it *is* rational for agents to be concerned with the beliefs of others.[11] Moreover such an analysis may clarify how it is that agents perceive the correct reduced form equation for price; it is clear from (20) that the distribution of P_t is nonstationary, so the usual stories may seem less convincing here. On the other hand there remains the question of whether a set of Nash equilibrium strategies could be attained by a process of learn-

[11] From a technical point of view the search for a rational expectations equilibrium was facilitated by the formulation of the problem in the strategy space. However, the equivalence of the two equilibrium concepts is not established, for suppose that θ were known by each agent and that $a > b$. Then there exists a rational expectations Equilibrium — it is the trivial self-fulfilling equilibrium. But Equation (10) is not well defined.

ing. In this regard an alternative equilibrium concept only leads to an old question in a new guise.

6. CONVERGENCE WITH NOISY AND BIASED MARKET SIGNALS: AN ALTERNATIVE APPROACH

This section presents an alternative approach to convergence to the self-fulfilling equilibrium. A model is proposed in which firms receive noisy and biased signals of market anticipations. Under apparently weak conditions firms eventually identify with some noise the true reduced form equation for the price.

It is assumed that each firm knows the parameters a and b and the distribution of ε_t. Each firm is completely ignorant of the way other firms form expectations. However before the output decision q_t each firm observes $\int E_{jt}(P_t)dj$ with some bias and some noise. In particular let

$$I_{it} = \int E_{jt}(P_t)dj + \Psi_i + \xi_{it} \tag{21}$$

where $\xi_{it} \sim N(0, \sigma_\xi^2)$. Parameters Ψ_i and θ are unknown. Firm i has a prior distribution on (Ψ_i, θ) at time t which is bivariate normal with mean $[E_{it}(\Psi_i), E_{it}(\theta)]$ and covariance matrix with k-th row n-th column element σ_{knt}. ε_t, ξ_{it}, and (θ, Ψ_i) are all independent. P_t and I_{it} are observed. S_t, ξ_{it}, and ε_t are unobserved. From (21), $S_t = a[I_{it} - \Psi_i - \xi_{it}]$. Market clearing yields

$$bP_t = \theta + \varepsilon_t - a[I_{it} - \Psi_i - \xi_{it}]\,. \tag{22}$$

Then for firm i

$$E_{it}(P_t) = \frac{E_{it}(\theta) - aI_{it} + aE_{it}(\Psi_i)}{b} \tag{23}$$

so that $\tilde{q}_{it} = (a/b)[E_{it}(\theta) - aI_{it} + aE_{it}(\Psi_i)]$. It is assumed that (23) is consistent with (21) in that integration over firms in (23) yields the term $\int E_{jt}(P_t)dj$ appearing in the right side of (21).

From (22) the random variable $(bP_t + aI_{it})$ is viewed by firm i as drawn from a normal distribution with unknown mean $\theta + a\Psi_i$ and specified variance $\sigma_\varepsilon^2 + a^2\sigma_\xi^2$. The prior of firm i on the unknown mean is normal with mean $m_{it} = E_{it}(\theta) + aE_{it}(\Psi_i)$ and variance $\sigma_{22t} + a^2\sigma_{11t} + 2a\sigma_{12t}$. It follows that $m_{it} \to \theta + a\Psi_i$ as $t \to \infty$ for each $i \in I$. Over time each firm learns the true value of $\theta + a\Psi_i$. Hence $bE_{it}(P_t) \to \theta - a\int E_{jt}(P_t)dj - a\xi_{it}$ as $t \to \infty$. Then under some regularity conditions as $t \to \infty$

$$\int E_{jt}(P_t)dj \longrightarrow \frac{\theta}{a+b} + \frac{a\int \xi_{jt}dj}{a+b}\,.$$

On average, in the limit, the self-fulfilling equilibrium (ii) is attained.

It should be stressed that the convergence in this sense does not require that the values of θ and Ψ_i each be known. For example, firm i may over-estimate the value of the parameter θ and under-estimate the value of the supply signal bias Ψ_i in such a way as to yield the true value of the linear combination $\theta + a\Psi_i$. In the limit, firm i can correctly estimate the relevant reduced form equation

$$bP_t + aI_{it} = \theta + \varepsilon_t + a\Psi_i + a\xi_{it} \tag{24}$$

though the structural equations (21) and $D_t = \theta - bP_t + \varepsilon_t$ remain under-identified. Of course $E_{it}(P_t)$ as determined from (24) completely determines the action of firm i.

7. CONCLUDING REMARKS

This paper may be regarded in part as an attempt to contribute to an understanding of the stability of rational expectations equilibria. In particular the stability of a self-fulfilling equilibrium in a class of models was examined. Consistent extended models were constructed, and the sequences of Nash equilibria of these extended models were analyzed. In focusing on consistent extended models and the Nash equilibrium concept, it became natural to examine the amount and nature of information required by agents in order for their decision problems to be well defined and for the sequence of Nash equilibria to convergence to a self-fulfilling equilibria. An analysis of the stability of a self-fulfilling equilibrium should naturally involve such questions.

Thus the stability of the self-fulfilling equilibrium is established, at least for some models. However, in a larger sense, questions concerning the stability of the more general notion of rational expectations equilibria may be ill-posed. One may of course analyze the sequence of equilibria of an extended model and perhaps establish convergence. But if the extended model is well defined its equilibria will necessarily be rational. Thus Cyert and DeGroot [1974] note that what they refer to as consistent models necessarily display rational expectations. And Section 5 of this paper makes the point that Nash equilibria in the strategy space may also be viewed as rational expectations equilibria with respect to price distributions conditioned on past observations.

This paper may also be regarded as an attempt to incorporate various psychological theories of economic behavior into a model in which agents are rational. It is argued that a concern with the beliefs of others is consistent with rationality, but economic considerations are also important. In Section 3 the terms $m_t(i)$ and $\int m_t(j)dj$ both enter into a maximizing decision rule for agent i. In Section 4 agents are concerned with expectations of all orders! Economic considerations may dominate in that in the limit agents learn the true value of θ and acquire uniform beliefs. In Section 6, I_{it} as a noisy and biased signal of $\int E_{jt}(P_t)dj$ is crucial to agent i. Again, in the limit, economic considerations dominate though the noise of the signals persists.

The model also points to an explanation for market surveys of intentions or anticipations. Such signals, even if noisy and biased, provide information as to the beliefs of others. This information may be crucial in making well defined the decision problem of the individual, in reducing uncertainty, and in establishing the speed of convergence to a stationary state. (In a more elaborate model, the quantity and quality of information to be disseminated would be determined endogenously.) Moreover the results of Section 6 establish that convergence to a self-fulfilling equilibrium is possible even if agents are not very well informed. Firms may be completely ignorant of the way others form expectations, and the bias $|\Psi_i|$ and noise σ_ξ^2 may be large. Though agents never identify the underlying structural relations of the model, they can learn the correct reduced form equation for price, the crucial variable in their decision problem, by simple Bayesian procedures.

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