A Market-Based Solution for Fire Sales and Other Pecuniary Externalities*

Weerachart T. Kilenthong[†] and Robert M. Townsend[‡]

Abstract

Pecuniary externalities are removed with market exchanges that internalize agent types influence on prices. Agents choose in the first period one price from among various possible prices they want to prevail in the future. Crucially, the right to trade in each and every exchange is priced. The total amount paid or compensation received is determined by the type specific factors which a type contributes to the chosen price collectively as a group, the units of their known excess demand function at that chosen price evaluated at their observed saving decision, times a common per unit price, which varies across exchanges.

Keywords: price externalities; Walrasian equilibrium; markets for rights to trade; market-based solution; collateral; exogenous incomplete markets; fire sales.

*We would like to thank Anmol Bhandari, Daniel Ehrlich, Masao Fukui, Mikhail Golosov, Lars Hansen, Bengt Holmstrom, Edward S. Prescott, Thomas Sargent, Alp Simsek, Harald Uhlig, Randall Wright, Haoxiang Zhu, the editor and anonymous referees, and seminar participants at conferences. Sartja Duangchaiyoosook provided excellent research assistance. Tee Kilenthong is grateful to financial support from UTCC. Robert Townsend is grateful to financial support from the NICHD under grant R01 HD027638, the CEPR and the DFID under grant MRG002_1255, the John Templeton Foundation, and the CFSP at the University of Chicago through a grant from the Bill & Melinda Gates Foundation. The findings and conclusions contained in the report are those of the author(s) and do not necessarily represent the views of the funders.

[†]University of the Thai Chamber of Commerce.

[‡]Massachusetts Institute of Technology.

1 Introduction

Both developed and emerging economies have experienced episodes of rapid credit expansion followed, in some cases, by a financial crisis, with a collapse in asset prices, credit, and investment. There is also a literature on fire sales in financial markets, e.g., Gorton and Metrick (2012). However, as Lorenzoni (2008) emphasizes, if the private sector had accurate expectations and correctly incorporated risk in its maximizing decisions, yet still decided to borrow heavily during booms, it means that the expected gain from increased investment more than compensated for the expected costs of financial distress. Thus one needs to understand how, and under what conditions, this private calculation leads to inefficient decisions at the social level. What is the externality? Likewise, how can it be remedied? Do we need regulation and government intervention, or can innovative market structure that internalizes the externality solve the problem.

There is a literature in the wake of the U.S. financial crisis that has focused on pecuniary externalities as the source of the problem. This literature seeks policy interventions and regulations to remedy the associated distortions, e.g., balance sheet effects, amplifiers and fire sales. Under pecuniary externalities, trading on a market adversely affects others via the revaluation of traded items. Solutions range from regulation of portfolios, restrictions on saving or credit, interest rate restrictions, fiscal policy, or taxes and subsidies levied by the government. However, general equilibrium theory suggests in other contexts that bundling, exclusivity and suitably designed additional markets for the objects associated with externalities could internalize those externalities, without the need of further policy interventions, or the need to quantify interventions, as the latter requires yet more information. Ex ante competition and equilibrium with market-determined prices for rights to trade in these additional markets can achieve a constrained-efficient allocation. Here we follow general equilibrium theory and remove pecuniary externalities in this way.

The influence of prices which can cause inefficiencies is akin to pollution, which has a remedy in competitive markets for the rights to pollute. We will draw an analog between pollution and price externalities to explain what we do. Specifically, for pollution, consider an initial economy with two goods, one period, one representative price taking consumer and one representative price taking firm. The consumer is endowed with one good which can be consumed or used by firms to produce the second good which the household also values. However, that production comes with air or water pollution, which gives the household disutility. The competitive equilibrium in which this pollution is not priced is not at a social optimum; marginal rates of substitution in consumption and production do not line up, as they would in the planner's problem. But now suppose we create markets in the rights to pollute. Factories have to buy rights to emit pollution, a cost which lowers their profit. They choose how much to produce and how much to pollute, consistent with permits purchased. Households sell rights to suffer pollution, a revenue added to their budget, and choose how much pollution they want, and how much to consume of the two goods. In the new decentralized market equilibrium, the demand for rights to pollute by firms and the supply of rights to suffer pollution by households will be equated by the appropriate price of rights, and money in equal amounts changes hands. The new equilibrium is Pareto optimal; it has some but less pollution. Of course, there needs to be some enforcement. Firms cannot pollute beyond rights purchased, as in cap and trade. The difference between cap and trade and the full market solution described here is that the quantity of permits is market determined and not fixed by the government.

Now consider an analogue economy with two goods, two periods, and two representative price-taking households, types 1 and 2. There is no uncertainty. The two decisions are intertemporal decisions, over time, and within-period decisions, across the two goods. Further, suppose that if there were no obstacles to trade and if markets were complete, the environment is such that, in a competitive equilibrium, type one would be a lender and type two a borrower. However, suppose in contrast that borrowing cannot happen in equilibrium.[1] Further, only one of the two goods can be stored, good 2. The relatively rich type 1 household ends up smoothing consumption over time on its own, not by lending to type 2 but by saving good 2. As a result, the price of the storage good 2 is low in the second period, as type 1 sells good 2 in the spot market then. This relative price is moving with

[1] This can be derived from limited commitment that the type 2 borrowers' promise to pay must be backed by collateral held in storage. By assumption the would-be borrower type 2 has little of the collateral good, and so the competitive equilibrium with limited commitment has no borrowing and lending.

saving, but both types take equilibrium prices as given. This relative price is source of a pecuniary externality.

The stored good in this intertemporal example above is like the input good in the initial pollution example. As with the pollution example where we needed markets for rights to pollute, here with this pecuniary externality, we need markets for rights to trade at the future relative price. Agent types need to choose in the first period the relative price at which they want to trade in the second period. That is, agents choose in the first period one price from among various possible prices they want to prevail in the future. One can think of an exchange or trading house earmarked by each possible such price. Each agent can choose the one it wants without regard to what any other agent is doing. But crucially, the right to trade in each and every exchange is priced. An agent type will pay or be compensated according to the price exchange chosen. The fee structure has a per unit price and quantity decomposition. Though any given exchange has a common per unit participation price, the total amount paid or compensation received is determined by the type specific factors which a type contributes to the chosen price collectively as a group, the units of their known excess demand function at that chosen price evaluated at their observed saving decision, again times a common per unit price. Again, the fees of every exchange are determined in this way, with the per unit price varying across exchanges, and of course the excess demand and savings moving as the relative spot price is moving across the exchanges. The fees of exchanges are the market determined decentralized feature. A tax subsidy alternative would require advance knowledge of these. See online Appendix C.

In our example economy, the saver type 1 will choose one of these price exchanges ex ante in the first period and has to acquire there rights to sell in the second period its savings of good 2, its excess supply, starting from its designated savings position. Type 1 subtracts the amount paid from the first period budget. Type 2 will also choose one of these price exchanges ex ante in the first period and has to sell rights there in the first period to buy the storage good 2 in the second period, starting from its zero-saving position. Type 1 adds revenue to the first period budget. In the decentralized competitive equilibrium, at the equilibrium fee structure, both types choose the same exchange in the first period, indexed by the same relative price, and the demand for rights of type 1 and supply of rights by type 2 are equated there. Likewise, by construction, the corresponding future spot market clears at the chosen price. The saver type 1 decides on storage of good 2, a constrained-optimal allocation is achieved, and there is some but less physical storage. As with pollution where we required enforcement to the level of pollution chosen, here we need enforcement of exclusivity, to ensure the price chosen is the price used. That is, an agent chooses only one exchange in the first period and cannot trade in multiple exchanges. Likewise, all spot trade in the second period must take place in the spot markets at the associated designated relative price that agents have chosen, showing up and displaying rights to trade there, and agents cannot make spot exchanges on the side.

Our analysis extends well beyond the example, which is intended to be illustrative. In our more general setup any agent type can make a promise to deliver in the future, but all promises must be backed by sufficient collateral so that promises can be honored. We can allow uncertainty about future states of the world, and promises can be state-contingent, as would be the collateral constraints, holding state by state to ensure state-contingent promises are honored. We can also allow exogenously incomplete markets. With incomplete security markets we can drop the collateral requirement, but generically competitive equilibrium are inefficient due to pecuniary externalities when there are multiple goods in spot markets. Trades in securities markets determine the distribution of income in spot markets, but by definition, when markets are incomplete, there is no way to hedge the associated income movements across states (e.g., Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986). In this case, as security positions move relative prices in multiple future states, rights are naturally a vector of rights over future states. We can remedy the pecuniary externality, so that allocations are constrained-efficient, though still not complete.[2] Other environments include a fire sale economy (Lorenzoni, 2008) as per our introduction to this paper, and a liquidity-constrained economy (Hart and Zingales, 2013), where there is too much saving. We also extend our method to environments with information imperfections, namely a moral hazard contract economy with multiple goods and retrade in spot markets (e.g., Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011) and a Diamond-Dybvig economy where

[2]Our solution is not about completing markets but remedying price externalities. See an example in the online Appendix E.

an agent's excess demands in interim bond markets is not known ex ante as each agent is subject to unobserved preference shocks that determine the direction of trade (e.g., Diamond and Dybvig, 1983; Jacklin, 1987). The key which allows this extension to private information about excess demand comes of course from incentive compatibility conditions on contracts. For a more complete description of these extensions with notation, see the online Appendix G. Our solution works in general as price is a "sufficient statistic" for the source of the problem, regardless of the underlying environment.

Our contribution is related to Coase (1960) in its emphasis on rights. Our pollution example is one of his lead examples. However, the Coase theorem is about how any given initial arbitrary distribution of rights would not matter if there were bargaining and no trading frictions, just as the initial allocation of rights to pollute in cap and trade would not matter, as efficiency works through opportunity costs.[3] In contrast, for us, rights are market determined. Thus, closer to what we do is the work of Arrow (1969), following Meade (1952), on the equivalence of solutions to planning problems and competitive equilibria with rights to trade in the objects causing non-pecuniary externalities. Keys are additional markets and excludability.[4]

The remainder of the paper proceeds as follows. Section 2 presents the saving economy to illustrate the ingredients. A general economy is described and the welfare theorems and existence theorem are stated in Section 3. Section 4 concludes. Appendix A presents the proof of the second welfare theorem, and additional results are in the online Appendices.

[3]We can relate our solution to Lindahl (1958) who uses agent specific prices to solve a public goods problem when there is heterogeneity in willingness to pay. Though the per unit price of an exchange is common, type specific excess demands make the total fee agent specific.

[4]See Chapter 11 of Mas-Colell et al. (1995) for more about this distinction between Coase (1960) and Arrow (1969). Interestingly, Arrow (1969) is less concerned about excludability, an intrinsic part of creating the necessary markets, as he feels this has a natural counterpart in many real world problems. Arrow (1969) is more concerned about the obvious small numbers problem. However, this part is easy to remedy with a continuum of traders and positive mass of each trader type.

2 A Saving Economy Illustrative of the Key Ingredients

This section features in notation the example economy of the introduction, a saving economy with no uncertainty. There are two periods, $t = 0, 1$. So this is a pure intertemporal economy. We thus make the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades in financial options. Indeed, in this example economy no securities will be traded in equilibrium, and in this way we focus on the market for rights to trade in spot markets, only, to remove the externality.

There is a continuum of agents of measure one. The agents are however divided into two heterogeneous types, $h = 1, 2$. Each type h consists of α^h fraction of the population. There are two consumption goods, which can be traded and consumed in each period. The two underlying goods are called good 1 and good 2. Good 1 cannot be stored (is completely perishable), while good 2 is storable from $t = 0$ to $t = 1$. Each unit of good 2 stored will become R units of good 2 at date $t = 1$. Let $k^h \in \mathbb{R}_+$ denote the saving (equivalent to the holding of good 2) of an agent type h at the end of period $t = 0$ to be carried to period $t = 1$. Let good 1 be the numeraire good in every date. The price in terms of the numeraire at which the good 2 can be sold in spot markets at $t = 1$ is the key object associated with the pecuniary externality. The contemporary preferences of agent type h are represented by the utility function $u^h\left(c_1^h, c_2^h\right)$, which is continuous, strictly concave, strictly increasing in both consumption goods, and satisfies the usual Inada conditions. Each agent type h is endowed with good 1 and good 2, $\mathbf{e}_t^h = \left(e_{1t}^h, e_{2t}^h\right) \in \mathbb{R}_+^2$ in period $t = 0, 1$.

For the numerical example we shall suppose each of the two types has an identical constant relative risk aversion (CRRA) utility function $u^h(c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}$, $h = 1, 2$. The endowment profiles are such that an agent type 1 is well endowed with 3 units of both goods in period $t = 0$ relative to one unit of both at $t = 1$, and vice versa for type 2. Each type h consists of $\frac{1}{2}$ fraction of the population, i.e., $\alpha^h = \frac{1}{2}$. In a full unconstrained optimum, with $R = 1$, each agent would consume 2 units of each good in each period.

2.1 Competitive Equilibrium with Externalities

We now are ready to formally define the competitive equilibrium as follows.

Definition 1. A competitive equilibrium is a specification of prices of good 2 in period $t = 0$ and $t = 1$, p_0 and p_1, respectively; consumptions $\left(c_{10}^h, c_{20}^h\right)$ at $t = 0$, trades $\left(\tau_{11}^h, \tau_{21}^h\right)$ at $t = 1$, and saving k^h decision made at $t = 0$ for each type h such that (i) agent type h solves

$$\max_{c_{10}^h, c_{20}^h, k^h, \tau_{11}^h, \tau_{21}^h} u^h\left(c_{10}^h, c_{20}^h\right) + u^h\left(e_{11}^h + \tau_{11}^h, e_{21}^h + Rk^h + \tau_{21}^h\right) \tag{1}$$

subject to the budget constraint in period $t = 0$, $c_{10}^h + p_0\left(c_{20}^h + k^h\right) \leq e_{10}^h + p_0 e_{20}^h$; the budget constraint in period $t = 1$, $\tau_{11}^h + p_1\tau_{21}^h = 0$; the non-negative saving constraint, $k^h \geq 0$; and (ii) the market for good 1 at $t = 0$ clears, $\sum_h \alpha^h c_{10}^h = \sum_h \alpha^h e_{10}^h$; the market for good 2 at $t = 0$ clears, $\sum_h \alpha^h\left[c_{20}^h + k^h\right] = \sum_h \alpha^h e_{20}^h$, and the markets for trades at $t = 1$ clear, $\sum_h \alpha^h \tau_{\ell 1}^h = 0, \forall \ell = 1, 2$.

Necessary conditions for competitive equilibrium related to saving k^h are as follows.

$$p_0 = \frac{u_{20}^h}{u_{10}^h} = \frac{u_{21}^h}{u_{10}^h}R + \frac{\eta_{nn}^h}{u_{10}^h}, \forall h = 1, 2. \tag{2}$$

where $u_{\ell t}^h \equiv \frac{\partial u^h\left(c_{1t}^h, c_{2t}^h\right)}{\partial c_{\ell t}}$ for $t = 0, 1$, and η_{nn}^h is the Lagrange multiplier for the non-negative saving constraint for an agent type h. Marginal rates of substitution and Euler equations are equated over agent types. There are two good with good 1 numeraire. The price of good 2 at $t = 0$, p_0, is derived from two components: (1) the value from the return R in the next period, and (2) the value from the fact that constrained agents would like to borrow but cannot, as saving cannot be negative.

To be more explicit about the spot market with price p_1 at $t = 1$, an agent type h with saving k^h is free to choose spot trades $\left(\tau_{11}^h, \tau_{21}^h\right)$ to solve the following utility maximization

$$V_1^h\left(k^h, p_1\right) = \max_{\tau_{11}^h, \tau_{21}^h} u^h\left(e_{11}^h + \tau_{11}^h, e_{21}^h + Rk^h + \tau_{21}^h\right) \tag{3}$$

subject to the spot market budget constraint at $t = 1$. Of course, to be consistent, the excess demands for good 1, the $\tau_{11}^{h*}\left(k^h, p_1\right)$ over types h, must satisfy the market-clearing conditions for trades at $t = 1$.[5] In fact, the spot market equilibrium price p_1 can be defined as a function

[5] With Walras' law, excess demands for good 2 are implied.

of savings of both types (keeping endowments implicit): $p_1 = p_1(k^1, k^2)$. This condition later in the planner's problem will be called the consistency constraint. These excess demands are crucial objects in the planner problem below and in the associated competitive equilibrium with rights.

Table 1: Equilibrium allocations with and without externalities.

	equilibrium with the externality (ex)						equilibrium with rights to trade (op)						
	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	$U^h(\mathbf{c}^h)$	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	$\Delta^h(p_1^{op})$	$U^h(\mathbf{c}^h)$
$h=1$	1.36	2.69	1.78	1.33	1.78	-2.2527	1.18	2.61	1.84	1.30	1.68	0.30	-2.2934
$h=2$	0	1.31	0.87	2.67	3.58	-2.5724	0	1.39	0.98	2.70	3.50	-0.30	-2.3904

For the numerical example, we summarize the equilibrium allocation in Table 1 featuring saving k^h and consumption $c_{\ell t}^h$. See online Appendix F.5.1 for the numerical derivation. Note that the first-best allocation features no saving, $k^1 = k^2 = 0$, and non-time-varying prices of good 2, $p_0 = p_1 = 1$. The first-best allocation suggests that agent 2 would like to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow and therefore will be constrained. The equilibrium with the externality present will have agent type 2 borrowing nothing and only trading in spot markets. Agent type 1 will be saving on its own to smooth consumption over time. With the externality (denoted "ex"), the price of good 2 in period $t = 0$ is $p_0^{ex} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and at date 1 is $p_1^{ex} = 0.5570$. Note that the price of good 2 is high at $t = 0$ since much is put into storage and likewise the price of good 2 is low at $t = 1$ when the storage is sold on the market. The right side of Table 1 with right will be determined below.

2.2 The Planner Problem

The planner assigns each agent type h to an allocation indexed by spot price p_1. That is, it is as if the planner is assigning type h to p_1-exchange or platform at $t = 0$ and assigns rights there, along with corresponding saving and $t = 0$ consumptions, with execution of trade in the corresponding p_1-exchange in the spot market at $t = 1$. Let $\delta^h(p_1)$ be the indicator variable for type h which is equal to 1 for the chosen p_1 and is zero otherwise. If $\delta^h(p_1) = 1$, then planner's choices for consumption and saving at $t = 0$ for type h are $\mathbf{c}_0^h(p_1)$

and $k^h(p_1)$, respectively, and trades at $t=1$ are $\boldsymbol{\tau}^h(p_1)$. If $\delta^h(p_1)=0$, then the rest of those p_1 conditional choices need not be specified, as agent h is not trading at p_1 but we write it all out to be clear. There is a way to economize on notation which we exploit in the appendix.

The planner must ensure that the pre-specified price p_1 is consistent with savings of both types so that price p_1 will be realized, i.e., $\left[\sum_h \delta^h(p_1)\right]\left[p_1 - p_1\left(k^1(p_1), k^2(p_1)\right)\right] = 0, \forall p_1$. These consistency constraints tie the pre-specified prices p_1 to the savings choices such that the pre-specified price is indeed the spot market equilibrium given the savings of both types. Equivalently, we can replace those conditions with the following consistency constraints.

$$\sum_h \alpha^h \delta^h(p_1)\, \Delta^h\left(k^h(p_1), p_1\right) = 0, \forall p_1, \tag{4}$$

where the rights to trade in exchange p_1 for an agent type h whose saving is k^h is assigned by the planner at $t=0$ but must be consistent with the excess demand for good 1 at $t=1$ for an agent type h with saving $k^h(p_1)$ and facing spot price p_1. That is, rights are defined by $\Delta^h\left(k^h(p_1), p_1\right) \equiv \tau_{11}^{h*}\left(k^h(p_1), p_1\right)$, defined earlier under equation (3).

The planner is aware of excess demands and that they must sum to zero for any p_1. Constraints (4) hold for all active and inactive markets at prices p_1. This control of the equilibrium prices p_1 is done by consistent choices of the storage k's. For an inactive market at a price p_1, the planner cannot assign both agents there and make the Pareto weighted sum of utilities higher given the solution to the planner problem below, but he could achieve that if there were some tiny slack in the constraint (4) that excess demands sum to zero. That is why constraints (4) are binding and give us the shadow prices of the rights to trade for each possible pre-specified price p_1, denoted by $\mu_\Delta(p_1)$. In summary, let $\mathbf{x}^h = \left[\mathbf{c}_0^h(p_1), k^h(p_1), \boldsymbol{\tau}^h(p_1), \delta^h(p_1), \Delta^h\left(k^h(p_1), p_1\right)\right]_{p_1}$ denote a typical bundle or allocation assigned to an agent type h running over all p_1.

The Pareto program with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h\right]_h} \sum_h \lambda^h \alpha^h \sum_{p_1} \delta^h(p_1)\left[u^h\left(c_{10}^h(p_1), c_{20}^h(p_1)\right) + u^h\left(e_{11}^h + \tau_{11}^h(p_1), e_{21}^h + Rk^h(p_1) + \tau_{21}^h(p_1)\right)\right]$$

subject to non-negative constraints on saving, the resource constraints for good 1 and good

2 at $t = 0$, the resource constraints for spot trades, the spot market budget constraints,

$$\sum_{p_1} \delta^h(p_1)\, k^h(p_1) \geq 0, \forall h = 1, 2, \tag{5}$$

$$\sum_h \sum_{p_1} \alpha^h \delta^h(p_1)\, c_{10}^h(p_1) = \sum_h \alpha^h e_{10}^h, \tag{6}$$

$$\sum_h \sum_{p_1} \alpha^h \delta^h(p_1) \left[c_{20}^h(p_1) + k^h(p_1)\right] = \sum_h \alpha^h e_{20}^h, \tag{7}$$

$$\sum_h \delta^h(p_1)\, \alpha^h \tau_{\ell 1}^h(p_1) = 0, \forall \ell = 1, 2; p_1, \tag{8}$$

$$\sum_{p_1} \delta^h(p_1) \left[\tau_{11}^h(p_1) + p_1 \tau_{21}^h(p_1)\right] = 0, \forall h = 1, 2, \tag{9}$$

and the consistency constraints (4).

The necessary conditions[6] for constrained optimality that are comparable to the one of a competitive equilibrium (2) are given by

$$p_0 = \frac{u_{20}^h}{u_{10}^h} = \frac{u_{21}^h}{u_{10}^h} R + \frac{\mu_{nn}^h}{\lambda^h \alpha^h u_{10}^h} - \frac{\mu_\Delta(p_1)}{\mu_{10}} \Delta_k^h\left(k^h(p_1), p_1\right), \forall h = 1, 2. \tag{10}$$

where the derivative of rights $\Delta_k^h\left(k^h, p_1\right) \equiv \frac{\partial \Delta^h\left(k^h, p_1\right)}{\partial k}$, μ_{10} are Lagrange multipliers for the resource constraints for good 1 in period $t = 0$, μ_{nn}^h are Lagrange multipliers for the non-negative saving constraints, and $\mu_\Delta(p_1)$ are the key Lagrange multipliers for the consistency constraints (4). Note that the solutions to the planner problem vary with Pareto weights λ^h, tracing out all possible Pareto optimal allocations.

Intuitively, the value or shadow price of good 2 consist of three components: (1) the value from the return R in the next period, (2) the value from the fact that an agent might be borrowing constrained, and (3) the (negative) value from its effect on the future $t = 1$ price through the impact of savings on the rights. Buying rights to buy good 1 at $t = 1$ is equivalent to selling good 2, a bad thing as oversaving is part of the distortion. The first two components were there also in the competitive equilibrium but the third here is new, though

[6]Given that the constraint set is not convex, the optimality conditions are necessary but may not be sufficient. This does not cause any problem to our externality argument, as for that part we simply need to show that an equilibrium cannot be constrained optimal, i.e., does not satisfy the necessary optimal conditions (10). We overcome the non-convexity problem using a mixture representation as in the appendix, where first-order conditions are necessary and sufficient.

quite natural for the planner. It reduces the incentive to save. As a result, the optimal level of saving is lower than the equilibrium level. This is formally proved in online Appendix F.3.

2.3 The Decentralization with Rights to Trade Δ^h and Prices P_Δ

The exchanges as trading houses or platforms are the same as in the planner problem. Let p_0 denote the price of good 2 in period $t = 0$, p_1 denote the price of good 2 in period $t = 1$, and $P_\Delta(p_1)$ denote the key price of the rights. All prices are taken as given. Agent type h is choosing the price and the objects conditional on the chosen price. Let the choice of exchange p_1 be described by indicators $\delta^h(p_1)$ and the bundles $\mathbf{x}^h = \left[\mathbf{c}_0^h(p_1), k^h(p_1), \boldsymbol{\tau}^h(p_1), \delta^h(p_1), \Delta^h\left(k^h(p_1), p_1\right)\right]_{p_1}$, including rights and excess demands. The excess demand Δ^h as function of p_1 and k^h is known and given but depends on the choice of k^h, which is endogenous. These Δ^h will also play the role of rights to enter designated exchanges at $t = 1$ and can in principle limit trade there. These objects Δ^h and $\mathbf{x}^h$ are the same objects as in the planner problem, but here the decision is decentralized at these given prices. The price of rights will correspond to the shadow price of rights in the planner problem for consistency constraint (4), and that is essentially why externalities are internalized. Trades are sequential over time. The initial endowments e_{10}^h and e_{20}^h at $t = 0$ are sold at consumption prices 1 and p_0 for goods 1 and 2, respectively, regardless of the exchange chosen. Then, in that chosen exchange, the rights $\Delta^h\left(k^h(p_1), p_1\right)$ with the indicated savings $k^h(p_1)$ determine the participation fee. That saving and consumption $\mathbf{c}_0^h(p_1)$ are then funded with type h purchasing these in the $t = 0$ spot market. We then move to the corresponding p_1 spot market at $t = 1$. Those rights Δ^h will be equal to what agent h will want to do at $t = 1$ since the rights are excess demands. So the solution is time consistent.[7]

Definition 2. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h\right]_h$, price of good 2 at $t = 0$, p_0, spot prices p_1 for active and potential spot markets at

[7]Equivalently, we could require that all trade in each of the exchanges be done with a central counter party, CCP, who becomes the buyer for every seller and the seller for every buyer. The CCP as a broker-dealer has to make sure that all trades clear and that saving and consumptions at $t = 0$ are funded. The continuum agent assumption removes any uncertainty. This equivalent formulation is useful when we have multiple active markets with the mixtures, as in the appendix.

$t = 1$, and the prices of the rights to trade $[\mathbf{P}_\Delta (p_1)]_{p_1}$ such that

(i) for any agent type h as a price taker, $\left[\mathbf{x}^h (p_1)\right]_{p_1}$ solves

$$\max_{\mathbf{x}^h} \sum_{p_1} \delta^h (p_1) \left[u^h \left(c_{10}^h (p_1), c_{20}^h (p_1) \right) + u^h \left(e_{11}^h + \tau_{11}^h (p_1), e_{21}^h + Rk^h (p_1) + \tau_{21}^h (p_1) \right) \right]$$

subject to the budget constraints in the first period, which now include prices of rights $P_\Delta (p_1)$ and demand for rights $\Delta^h \left(k^h (p_1), p_1\right)$

$$\sum_{p_1} \delta^h (p_1) \left[c_{10}^h (p_1) + p_0 \left[c_{20}^h (p_1) + k^h (p_1) \right] + P_\Delta (p_1) \Delta^h \left(k^h (p_1), p_1 \right) \right] \leq e_{10}^h + p_0 e_{20}^h, \quad (11)$$

and the spot-budget constraint in period $t = 1$ (9) and the non-negative saving constraint (5) for the agent type h,

(ii) market-clearing conditions (4), (6), (7), (8) hold.

Using the similar steps as in the preceding section, we can then write the necessary conditions as follows.

$$p_0 = \frac{u_{20}^h}{u_{10}^h} = \frac{u_{21}^h}{u_{10}^h} R + \frac{\eta_{nn}^h}{u_{10}^h} - P_\Delta (p_1) \Delta_k^h \left(k^h (p_1), p_1 \right), \forall h = 1, 2. \quad (12)$$

which is starkly similar to (10) of the planner and that is the entire point. The last two terms in both equations are the externality correction terms that come from incorporating rights and are similar to those of the planner problem. Indeed, they are identical when we match the Lagrange multipliers and prices from the planner problem and the new equilibrium using the following conditions: $P_\Delta (p_1) = \frac{\mu_\Delta (p_1)}{\mu_{10}}$ and $\eta_{nn}^h = \frac{\mu_{nn}^h}{\lambda^h \alpha^h}$. The first condition shows how to recover prices of the rights to trade in both active and inactive exchanges from the Lagrange multipliers for the consistency constraints. Note that the Pareto weights λ^h associated with the competitive equilibrium can be recovered using the following condition: $\frac{\lambda^h \alpha^h}{\mu_{10}} = \frac{1}{\eta_{bc,0}^h}$, where μ_{10} is the Lagrange multiplier on (6), and $\eta_{bc,0}^h$ is the Lagrange multiplier on (11). The competitive equilibrium with rights picks out one of the Pareto optimal allocation as a solution to the planner problem at Pareto weights λ^h which do not require lump-sum taxes and transfers.

For the numerical example, the competitive equilibrium with rights to trade has *one and only one active exchange*, $p_1^{op} = 0.5974$, even though all spot markets are available in principle a priori for trade. That is, in equilibrium, both types optimally choose to trade in the same p_1-exchange at $t = 0$ and hence the same p_1 spot market with, $p_1^{op} = 0.5974$. Note that as anticipated from the right hand side of Table 1, there is less saving with the externality corrected, so the spot price of good 2 is lower at $t = 0$. Therefore, the price of good 2 is higher at $t = 1$ relative to the equilibrium with externalities present: $0.5570 = p_1^{ex} < p_1^{op} = 0.5974$. A borrowing constrained type 2 agent is facing a higher price than at p_1^{ex}, though that price exchange is inactive in the new equilibrium, he could have traded there. That is why agent type 2 is compensated.

Table 2 presents equilibrium prices/fees of rights to trade, that is $P_\Delta(p_1)$ not only for p_1^{op} but also other, different spot price levels p_1. Note again that the prices/fees of non-active spot markets are available, but facing such prices, agents do not want to trade in them.[8] Again, both types choose $p_1^{op} = 0.5974$.

Table 2: Equilibrium prices of rights to trade in spot markets $P_\Delta(p_1)$ at price p_1.

	$p_1 = 0.5770$	$p_1^{op} = 0.5974$	$p_1 = 0.6181$
$P_\Delta(p_1)$	1.1383	1.2116	1.2840

An agent type 1 comes into the spot market at $t = 1$ with good 2 in storage. So, type 1 buys the right to buy good 1 and sell good 2 in amount $\Delta^1(p_1^{op}) = 0.2970$, where $\Delta^h(p_1^{op}) \equiv \Delta^h\left(k^h(p_1^{op}), p_1^{op}\right)$. This makes sense as agent type 1 is doing the saving in good 2 and there is too much saving in the (ex) equilibrium. On the other hand, an agent type 2 will be paid for her willingness to choose that market $p_1^{op} = 0.5974$. Agent type 2 is facing a higher price of the good 2, and good 2 will be purchased. But there is compensation. In particular, a constrained agent $(h = 2)$ with $\Delta^2(p_1^{op}) = -0.2970$ and $P_\Delta(p_1^{op}) = 1.2116$ is receiving $-P_\Delta(p_1^{op})\Delta^2(p_1^{op}) = 0.3598$ in period $t = 0$ for being in the spot market $p_1^{op} = 0.5974$. Graphically, this shifts her budget line outward at $t = 0$ by $T = 0.3598$, hence in the

[8]Prices used are based on shadow prices from the planner problem, hence true marginal costs. As in standard price theory, markets in some goods can be cleared at prices implying zero activity, as when prices at marginal costs are strictly larger than the willingness to pay. Equilibrium prices can be indeterminate in a certain range in the sense that marginal cost prices can be lowered a bit but not impact the allocation. We return to this issue in the appendix.

direction of being less constrained.[9]

3 General Economy

This section presents an extension of the leading example by adding uncertainty, traded securities, yet allowing for market incompleteness and collateral constraints.[10] Consider an economy with S possible states of nature at $t = 1$, i.e., $s = 1, \dots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. There are H types of the continuum of agents in the population with fractions $\alpha^h > 0$, for $h = 1, 2, \dots, H$ such that $\sum_h \alpha^h = 1$. Endowment profiles are e^h_{1s} and e^h_{2s} for goods 1 and 2, where for convenience of notation $s = 0$ is the endowment at date $t = 0$. The utility functions u^h are strictly concave with other regularity conditions. There are J securities available for purchase or sale at $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets at $t = 1$ where $D_{js} \in \mathbb{R}_+$ is the payoff of asset j in units of good 1 (the numeraire good) in state $s = 1, 2, \dots, S$. Here we do not include securities paying in good 2 as there is trade in the two goods in spot markets, so these are not needed. Let θ^h_j denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$ with $\boldsymbol{\theta}^h \equiv \left[\theta^h_j\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise. The collateral constraint in state s states that there must be sufficient collateral in value to honor all promises:

$$p_s R_s k^h + \sum_j D_{js} \theta^h_j \geq 0, \forall s. \tag{13}$$

Equation (13) can be rewritten with securities $\theta^h_j > 0$ as investments with payouts D_{js} added to the value of collateral in terms of the numeraire on the left hand side and the securities $\theta^h_j < 0$ as promises with obligations D_{js} on the right hand side. This is a generalized version of $k^h \geq 0$ in the saving economy. More general obstacle-to-trade constraints applicable to

[9]Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium. Thus if nothing else were done, internalizing the externality would be beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we state in Section 3.1 and prove in the appendix.

[10]We provide numerical examples of an economy with active security holdings and an economy with incomplete markets in the online Appendix F.5.3.

our market-based approach are presented in the online appendix.

With potentially incomplete security markets, a given security traded at $t = 0$ has implications in general for most if not all spot prices at $t = 1$. This is one source of externalities. In addition, promises are at least potentially backed by collateral good 2, which is carried over to $t = 1$, another source of externalities as in the saving example. To internalize these externalities, we thus need rights to trade indexed by the vector of spot prices $\mathbf{p} = [p_s]_s$ over all states s.[11] That is, each $\mathbf{p}$-exchange must naturally deal with S spot markets as a bundle. As a result, all objects are indexed by vector $\mathbf{p}$. This is where there is a subtle difference from the saving economy. Let $Q_j(\mathbf{p})$ denote the price of security j executed in an exchange $\mathbf{p}$ with vector $\mathbf{Q}(\mathbf{p}) \equiv [Q_j(\mathbf{p})]_j$.

The right to trade in an exchange $\mathbf{p}$ is denoted by a vector of rights $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta_s^h(\mathbf{p})\right]_s$, where for the component at state s,

$$\Delta_s^h(\mathbf{p}) \equiv \tau_{1s}^{h*}\left(p_s, k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p})\right), \forall s, \tag{14}$$

which is the standard excess demand for good 1 in the spot market s at $t = 1$ for an agent type h holding collateral $k^h(\mathbf{p})$, securities $\boldsymbol{\theta}^h(\mathbf{p})$, and being in an exchange $\mathbf{p}$. For brevity, we write this right $\Delta_s^h(\mathbf{p})$ as a function of the spot prices $\mathbf{p}$ only on the left hand side of (14) even though the excess demand depends on the pre-trade position coming from collateral/savings and securities. Note that we now make explicit both the endowments in state s and security holdings. Let $\mathbf{x}^h = \left[\mathbf{c}_0^h(\mathbf{p}), k^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \delta^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{\mathbf{p}}$ denote a typical bundle or allocation for an agent type h.

The Pareto program with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{[\mathbf{x}^h]_h} \sum_h \lambda^h \alpha^h \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u\big(c_{10}^h(\mathbf{p}), c_{20}^h(\mathbf{p})\big) + \sum_s \pi_s u\big(e_{1s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{1s}^h(\mathbf{p}), e_{2s}^h + R_s k^h(\mathbf{p}) + \tau_{2s}^h(\mathbf{p})\big)\right]$$

subject to collateral constraints, the resource constraints for good 1 and good 2 in period $t = 0$, the adding-up constraints for spot trades, the spot market budget constraints, the adding-up constraints for securities, and the consistency constraints, respectively,

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[p_s R_s k^h(\mathbf{p}) + \sum_j D_{js}\theta_j^h(\mathbf{p})\right] \geq 0, \forall h = 1, 2, \tag{15}$$

$$\sum_h \sum_{\mathbf{p}} \alpha^h \delta^h(\mathbf{p})\, c_{10}^h(\mathbf{p}) = \sum_h \alpha^h e_{10}^h, \tag{16}$$

[11]Henceforth, bold typeface refers to a vector.

$$\sum_{h}\sum_{\mathbf{p}}\alpha^{h}\delta^{h}(\mathbf{p})\left[c_{20}^{h}(\mathbf{p})+k^{h}(\mathbf{p})\right]=\sum_{h}\alpha^{h}e_{20}^{h}, \tag{17}$$

$$\sum_{h}\delta^{h}(\mathbf{p})\,\alpha^{h}\tau_{\ell s}^{h}(\mathbf{p})=0,\forall\ell=1,2;s;\mathbf{p}, \tag{18}$$

$$\sum_{\mathbf{p}}\delta^{h}(\mathbf{p})\left[\tau_{1s}^{h}(\mathbf{p})+p_{s}\tau_{2s}^{h}(\mathbf{p})\right]=0,\forall h=1,2;s, \tag{19}$$

$$\sum_{h}\delta^{h}(\mathbf{p})\,\alpha^{h}\theta_{j}^{h}(\mathbf{p})=0,\forall j;\mathbf{p}, \tag{20}$$

$$\sum_{h}\delta^{h}(\mathbf{p})\,\alpha^{h}\Delta_{s}^{h}(\mathbf{p})=0,\forall s;\mathbf{p}. \tag{21}$$

As illustrated in the example and formally proved in the following section, a constrained optimal allocation, a solution to the Pareto program, can be decentralized in a competitive equilibrium with rights to trade, a generalized version of the one defined in Section 2.3.

Definition 3. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^{h}\right]_{h}$, price of good 2 at $t=0$, p_0, spot prices $\mathbf{p}=\left[p_{s}\right]_{s}$ for active and potential spot markets at $t=1$, and the prices of securities and the rights to trade $\left[\mathbf{Q}(\mathbf{p}),\mathbf{P}_{\Delta}(\mathbf{p})\right]_{\mathbf{p}}$ such that

(i) for any agent type h as a price taker, $\left[\mathbf{x}^{h}(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\mathbf{x}^{h}}\sum_{\mathbf{p}}\delta^{h}(\mathbf{p})\left[u^{h}\left(c_{10}^{h}(\mathbf{p}),c_{20}^{h}(\mathbf{p})\right)+\sum_{s}\pi_{s}u^{h}\left(e_{1s}^{h}+\sum_{j}D_{js}\theta_{j}^{h}(\mathbf{p})+\tau_{1s}^{h}(\mathbf{p}),e_{2s}^{h}+R_{s}k^{h}(\mathbf{p})+\tau_{2s}^{h}(\mathbf{p})\right)\right]$$

subject to the budget constraints in the first period, which now includes securities $\boldsymbol{\theta}^{h}$ at vector of prices $\mathbf{Q}$ and vector of rights $\boldsymbol{\Delta}^{h}$ at vector of prices $\mathbf{P}_{\Delta}$

$$\sum_{\mathbf{p}}\delta^{h}(\mathbf{p})\left[c_{10}^{h}(\mathbf{p})+p_{0}\left[c_{20}^{h}(\mathbf{p})+k^{h}(\mathbf{p})\right]+\sum_{j}Q_{j}(\mathbf{p})\,\theta_{j}^{h}(\mathbf{p})+P_{\Delta}(\mathbf{p})\,\Delta^{h}(\mathbf{p})\right]\leq e_{10}^{h}+p_{0}e_{20}^{h}, \tag{22}$$

the spot-budget constraint in period $t=1$ (18), and the collateral constraints (15),

(ii) market-clearing conditions (16), (17), (18), (20), (21) hold.

3.1 Welfare Theorems and Existence Theorem

By suitable extension of the commodity space that allows mixture representations as formalized in the appendix, the economy becomes a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption sets are compact and convex, and the utility functions are linear.

Theorem 1. *Any Pareto optimal allocation corresponding with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with rights to trade with transfers.*

The standard proof applies. Any constrained optimal allocation can be decentralized as a compensated equilibrium matching up first-order conditions. Then, use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers.[12] See Appendix A.1.

Theorem 2. *With local non-satiation of preferences and positive endowments, a competitive equilibrium with rights to trade exists and is optimal.*

We use (Negishi, 1960)'s mapping method. The proof benefits from the second welfare theorem that the solution to the Pareto program is a competitive equilibrium with transfers. We then show that a fixed-point of the mapping exists and represents a competitive equilibrium without transfers and using the mapping is constrained optimal. See the online appendix for the proof.

In addition, we can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when markets for rights to trade are introduced. See more details in the online Appendix D.

4 Conclusion

Our solution concept extends to many other well-known environments in the literature that have prices in constraints beyond the role of prices in budget constraints, as happens in many models. The collateral constraints are featured in the general model, but more generally there are sets of obstacle-to-trade constraints, which include as arguments not only consumption, securities, spot trades, and inputs and outputs from production but also vectors of prices. In the online appendix, we write out these constraints for additional prototype economies mentioned in the introduction.

We conclude this paper with some comments on implementation of our solution to remedy pecuniary externalities. We do this, to be specific, in the context of markets that have

[12]If a competitive equilibrium exists, it can be shown to be optimal simply by the first welfare theorem.

suffered from fire sales, as in some markets for repo and credit default swaps, though we would need to change market structure, as we now outline below. Currently in these and other markets registered securities traders buy/sell securities and cash, but increasingly there are more centralized inter-dealer broker dealers, IDBs, that mediate this trade by running hybrid e-platforms. We would require for implementation that securities trades be done on these more centralized e-platforms, become the p-platforms of the theory. The requirement that trade be centralized is not new. In many instances, especially post-crisis with additional regulation, netting is required on centralized clearing platforms, an intermediary who is a buyer for every seller and a seller for every buyer, with collateral if any recorded and held in escrow and some fees charged.[13] Again these hybrid e-platforms and CCP platforms become the **p**-exchanges of the theory. As we require for implementation, it is already the case that securities are held, maintained, and registered on electronic book entry systems and direct transfers of securities are made through specified utilities.[14] Exchange and payment cannot be done outside these utilities and we would require that too. Further, it is not uncommon that only some are allowed to participate.[15] So, the necessary exclusivity required by the theory is not hard to imagine. For us this right to participate would be done with fees (or compensation) associated with purchased rights to trade (or sold rights) on the designated p-exchanges. Borrowers would also designate collateral as part of their excess demand, as is done now with repo. Currently, when a dealer borrower defaults on the repo, its investors receive the securities posted as collateral at prevailing spot prices. Related, a borrower may wish to get some of the securities used as collateral back during the trading cycle and in that case the clearing bank determines the cash value at current market prices, at that point in time. In contrast, we would require for implementation that clearing banks partition

[13] As a consequence of Dodd Frank regulation, market participants in CDS index contracts are required to trade in a Swap Execution Facility and to clear in a CCP platform. These platforms charge fees and commission based typically on volume.

[14] Example of utilities are Fedwire Securities Service and Fedwire Funds.

[15] Fed has a list of authorized broker dealers for the OTC treasury market. Mutual funds and other investors are not allowed to deal directly on Fedwire funds and Fedwire securities and go through broker dealers. More generally, some market exchanges are said to restrict access to high frequency traders. Exchange traded funds name a restricted set of Authorized Participants who are allowed to deal with the sponsor of the fund.

collateral consistent with exclusivity chosen by the trader and thus can unwind collateral at the pre-specified state-contingent prices agreed to by the trader at the time when the trade is entered into. To emphasize again, there would be a range of potential pre-specified prices for traders to choose from, varying in fees and compensation, and the price of rights in each p-exchange would be determined by some auction or other centralized trading platform.

Potential difficulties that will have to be thought through include incomplete enumeration of future states, in which case we hope our solution works as an approximation. Jeremey Stein has written about ex ante fees, a price based mechanism, for the use of a central bank credit liquidity facility, which might be contingent on some adverse states such as financial shocks, when liquidity is at a premium. Another difficulty would be vested interests that resist market reform without compensation, though this issue is not new or peculiar to the situation here. Finally, there could be a problem with inactive exchanges. The theory requires that traders can choose any p-exchange they want and we do not want the inactive ones to be eliminated prematurely. Ex ante we do not know which exchanges these will be. As Stein noted, the use of rights priced with fees in financial markets can help deal with situations in which even well informed regulators cannot know the exact requirements that might be needed.[16]

References

Acemoglu, D. and A. Simsek (2012). Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading. *Working Paper*.

Arrow, K. (1969). The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Nonmarket Allocation. *The Analysis and Evaluation of Public Expenditure: The PPB System 1*, 59–73.

Coase, R. H. (1960). The problem of social cost. *Journal of Law and Economics 3*, 1.

Debreu, G. (1954). Valuation Equilibrium and Pareto Optimum. *Proceedings of the National Academy of Sciences of the United States of America 40*(7), 588.

[16]See the article at https://www.federalreserve.gov/newsevents/speech/stein20130419a.htm.

Diamond, D. and P. Dybvig (1983). Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy 91*(3), 401.

Geanakoplos, J. and H. Polemarchakis (1986). Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete. *Uncertainty, Information and Communication: Essays in Honor of KJ Arrow 3*, 65–96.

Gorton, G. and A. Metrick (2012). Securitized banking and the run on repo. *Journal of Financial Economics 104*(3), 425–451.

Greenwald, B. and J. Stiglitz (1986). Externalities in Economies with Imperfect Information and Incomplete Markets. *Quarterly Journal of Economics 101*(2), 229–264.

Hart, O. D. and L. Zingales (2013). Inefficient Provision of Liquidity. Technical report, Working Paper.

Jacklin, C. (1987). Demand Deposits, Trading Restrictions, and Risk Sharing. *Contractual Arrangements for Intertemporal Trade*, 26–47.

Kilenthong, W. T. and R. M. Townsend (2011). Information-Constrained Optima with Retrading: An Externality and its Market-Based Solution. *Journal of Economic Theory 146*(3), 1042–1077.

Lindahl, E. (1958). Just Taxation–A Positive Solution. In R. A. Musgrave and A. T. Peacock (Eds.), *Classics in the theory of public finance*. Macmillan, London.

Lorenzoni, G. (2008). Inefficient Credit Booms. *Review of Economic Studies 75*(3), 809–833.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995). *Microeconomic Theory*, Volume 1. Oxford university press New York.

Meade, J. E. (1952). External Economies and Diseconomies in a Competitive Situation. *The economic journal*, 54–67.

Negishi, T. (1960). Welfare Economics and Existence of an Equilibrium for a Competitive Economy. *Metroeconomica 12*(2-3), 92–97.

Prescott, E. C. and R. M. Townsend (1984). General Competitive Analysis in an Economy with Private Information. *International Economic Review 25*(1), 1–20.

A Proofs

To overcome a potential non-convexity problem, we use the mixture representation for the proof, as in Prescott and Townsend (1984). Let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be the fraction of agents type h assigned to a bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. This x^h puts mass on the entire bundle including $\mathbf{p}$. This is a way we economize the notation relative to the discrete choice $\delta^h(\mathbf{p})$ of the text. This notation also allows a positive mass between zero and one, with more than one active exchange, as shown in a numerical example in online Appendix F.5.3. From the individual point of view, a non-degenerate x^h is a lottery but in the aggregate there is no uncertainty, which we utilize in the broker-dealer problem below.

At the individual level, for each agent type h, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0$ denote the probability of receiving period $t = 0$ consumption $\mathbf{c}_0$, collateral k, securities $\boldsymbol{\theta}$, period $t = 1$ spot trades $\boldsymbol{\tau}$, and being in exchanges indexed by $\mathbf{p} \equiv [p_s]_s$ with rights to trade $\boldsymbol{\Delta}$. We write again the spot market budget and the collateral constraints in state s:

$$\tau_{1s} + p_s \tau_{2s} = 0, \forall s, \tag{23}$$

$$p_s R_s k + \sum_j D_{js} \theta_j \geq 0, \forall s. \tag{24}$$

Accordingly, we impose the following condition on a probability measure:

$$x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0 \text{ if } (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \text{ satisfies (14), (23), (24),} \tag{25}$$

and zero otherwise. The consumption possibility set of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}^n_+ : \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) = 1, \text{ and (25) holds} \right\}. \tag{26}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option. For notational purposes, let $\mathbf{w} \equiv (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be a typical bundle, and the utility derived from it for an agent type h is defined by $U^h(\mathbf{w}) = u^h\big(c_{10}, c_{20}\big) + \sum_s \pi_s u^h\big(e^h_{1s} + \sum_j D_{js}\theta^h_j + \tau_{1s}, e^h_{2s} + R_s k + \tau_{2s}\big)$. Then, we have the maximization problem for agents as part of the definition of equilibrium: for each h, $\mathbf{x}^h \in X^h$ solves

$$\max_{\mathbf{x}^h} \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w}) \tag{27}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 0$ budget constraint, that the valuation of endowments sold provides revenue for purchase of the lotteries.

$$\sum_{\mathbf{w}} P(\mathbf{w})\, x^h(\mathbf{w}) \quad \leq \quad e^h_{10} + p_0 e^h_{20}, \tag{28}$$

taking price of good 2 at $t = 0$, p_0, and prices of lottery, $P(\mathbf{w})$ as given.

We introduce broker dealers that run the $\mathbf{p}$-platforms and deal with households for trades in securities, collateral, rights to trade and spot trades. The consumption $\mathbf{c}_0$ and collateral k commitments are sold but must be funded by the requisite amount of consumption goods and collateral. Securities, rights and spot trades do not require resources but are cleared by the broker-dealers. There are constant returns to scale in these activities so it is as if there were one representative broker-dealer. Let $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ denote the quantity of commitment to provide $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. See Prescott and Townsend (1984) for the introduction of broker-dealer. The broker-dealer takes prices $p_0, P(\mathbf{w})$ as given and supplies $\mathbf{b}$ to solve the following profit maximization problem:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} b(\mathbf{w}) \left[P(\mathbf{w}) - c_{10} - p_0 c_{20} - p_0 k\right] \tag{29}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \theta_j = 0, \ \forall j; \mathbf{p}, \tag{30}$$

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \tau_{\ell s} = 0, \ \forall s; \ell = 1, 2; \mathbf{p}, \tag{31}$$

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \Delta_s = 0 \ , \forall s; \mathbf{p}. \tag{32}$$

Market clearing conditions in the two consumption goods is standard, purchased consumptions and collateral by the broker-dealer equals supply of endowments from the households:

$$\sum_{\mathbf{w}} b(\mathbf{w})\, c_{10} = \sum_h \alpha^h e_{10}^h, \tag{33}$$

$$\sum_{\mathbf{w}} b(\mathbf{w})\, [c_{20} + k] = \sum_h \alpha^h e_{20}^h. \tag{34}$$

The net demand for contracts by households, allowing non-degenerate fractions in the population, equals the supply of contracts by the broker-dealer:

$$\sum_h \alpha^h x^h(\mathbf{w}) = b(\mathbf{w}),\ \forall \mathbf{w}. \tag{35}$$

See online Appendix F.5.3 for a particular clarified example of what broker-dealers in the context of an environment with multiple active exchanges.

Definition 4. A competitive equilibrium with rights to trade (with mixtures) is a specification of allocation $(\mathbf{x}^h, \mathbf{b})$, and prices $(p_0, P(\mathbf{w}))$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves the utility maximization problem (27) taking prices as given;

(ii) for the broker-dealer, $\mathbf{b}$ solves the maximization problem (29), taking prices as given;

(iii) market clearing conditions (33)-(35) hold.

The Planner Problem in the Mixture/Lottery Representation

The Pareto problem with Pareto weights $[\lambda^h]_h$ is defined as follows.

$$\max_{[\mathbf{x}^h]_h} \sum_h \lambda^h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w}) \tag{36}$$

subject to

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, c_{10} = \sum_h \alpha^h e_{10}^h, \tag{37}$$

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, [c_{20} + k] = \sum_h \alpha^h e_{20}^h, \tag{38}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \tau_{\ell s} = 0, \forall \ell; s; \mathbf{p}, \tag{39}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \theta_j = 0, \forall j; \mathbf{p}, \tag{40}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \Delta_s = 0, \forall s; \mathbf{p}. \tag{41}$$

A.1 Proof of The Second Welfare Theorem

Proof of Theorem 1. Since the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof is divided into three steps.

(i) Kuhn-Tucker conditions for a compensated equilibrium allocation: Let $\hat{\gamma}_U^h$ and $\hat{\gamma}_l^h$ be the Lagrange multiplier for the reservation-utility constraint, and for the probability constraint, respectively. The optimal condition for $x^h(\mathbf{w})$ is given by

$$\hat{\gamma}_U^h U^h(\mathbf{w}) \leq P(\mathbf{w}) + \hat{\gamma}_l^h, \tag{42}$$

where the inequality holds with equality if $x^h(\mathbf{w}) > 0$. The optimal condition for the intermediary's profit maximization problem implies that, for any typical bundle $\mathbf{w}$,

$$P(\mathbf{w}) \leq c_{10} + p_0[c_{20} + k] + \sum_j \widehat{Q}_j(\mathbf{p})\theta_j + \sum_s \sum_\ell \hat{p}_\ell(\mathbf{p}, s)\tau_{\ell s} + \sum_s \widehat{P}_\Delta(\mathbf{p}, s)\Delta_s, \tag{43}$$

where $\widehat{Q}_j(\mathbf{p})$, $\hat{p}_\ell(\mathbf{p}, s)$ and $\widehat{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (30)-(32). The condition holds with equality if $b(\mathbf{w}) > 0$.

(ii) Kuhn-Tucker conditions for Pareto optimal allocations: A solution to the Pareto program satisfies the following optimal condition

$$\lambda^h U^h(\mathbf{w}) \leq \tilde{p}_{10} + \tilde{p}_{20}[c_20 + k] + \sum_j \widetilde{Q}_j(\mathbf{p})\theta_j + \sum_s \sum_\ell \tilde{p}_\ell(\mathbf{p}, s)\tau_{\ell s} + \sum_s \widetilde{P}_\Delta(\mathbf{p}, s)\Delta_s + \tilde{\gamma}_l^h, \tag{44}$$

where $\tilde{\gamma}_l^h$ is the Lagrange multiplier for the probability constraint, and $\tilde{p}_{10}$, $\tilde{p}_{20}$, $\widetilde{Q}_j(\mathbf{p})$, $\tilde{p}_\ell(\mathbf{p}, s)$ and $\widetilde{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (37)-(41), respectively. Again, the condition holds with equality if $x^h(\mathbf{w}) > 0$.

(iii) Matching dual variables and prices: We can now set $\hat{\gamma}_U^h = \frac{\lambda^h}{\tilde{p}_{10}}$, $p_0 = \frac{\tilde{p}_{20}}{\tilde{p}_{10}}$, $\widehat{Q}_j(\mathbf{p}) = \frac{\widetilde{Q}_j(\mathbf{p})}{\tilde{p}_{10}}$, $\hat{p}_\ell(\mathbf{p}, s) = \frac{\tilde{p}_\ell(\mathbf{p},s)}{\tilde{p}_{10}}$ and $\widehat{P}_\Delta(\mathbf{p}, s) = \frac{\widetilde{P}_\Delta(\mathbf{p},s)}{\tilde{p}_{10}}$, and $\hat{\gamma}_l^h = \frac{\tilde{\gamma}_l^h}{\tilde{p}_{10}}$. These matching conditions imply that the optimal conditions of the Pareto program are equivalent to the optimal conditions for consumers' and market-maker's problems in the compensated equilibrium. To sum up, any Pareto optimal allocation is a compensated equilibrium.

We can show that any compensated equilibrium, corresponding to $\lambda^h > 0$, is a competitive equilibrium with transfers using the cheaper point argument, which is obvious given the strictly positive Pareto weight and strictly positive endowment. Using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. □