A Market-Based Solution for Fire Sales and Other Pecuniary Externalities*

Weerachart T. Kilenthong[†] and Robert M. Townsend[‡]

Abstract

In economies with a continuum of agents of different types, pecuniary externalities are removed with market exchanges. Agents choose from among various possible prices they want to prevail in the future and buy or sell rights in these market exchanges for future trade. Each agent can choose the exchange it wants without regard to what any other agent is doing. But crucially, the right to trade in each and every exchange is priced. The fee structure has a per unit price and quantity decomposition: a price, as determined by the exchange chosen, times the quantity of rights acquired.

Keywords: price externalities; Walrasian equilibrium; markets for rights to trade; market-based solution; collateral; exogenous incomplete markets; fire sales.

1 Introduction

Both developed and emerging economies have experienced episodes of rapid credit expansion followed, in some cases, by a financial crisis, with a collapse in asset prices, credit, and

*We would like to thank Anmol Bhandari, Daniel Ehrlich, Masao Fukui, Mikhail Golosov, Lars Hansen, Bengt Holmstrom, Edward S. Prescott, Thomas Sargent, Alp Simsek, Harald Uhlig, Randall Wright, Haoxiang Zhu, the editor and anonymous referees, and seminar participants at conferences. Sartja Duangchaiyoosook provided excellent research assistance. Tee Kilenthong is grateful to financial support from UTCC. Robert Townsend is grateful to financial support from the NICHD under grant R01 HD027638, the CEPR and the DFID under grant MRG002_1255, the John Templeton Foundation, and the CFSP at the University of Chicago through a grant from the Bill & Melinda Gates Foundation. The findings and conclusions contained in the report are those of the author(s) and do not necessarily represent the views of the funders.

[†]University of the Thai Chamber of Commerce.

[‡]Massachusetts Institute of Technology.

investment. There is also a literature on fire sales in financial markets, e.g., Gorton and Metrick (2012). However, as Lorenzoni (2008) emphasizes, if the private sector had accurate expectations and correctly incorporated risk in its maximizing decisions, yet still decided to borrow heavily during booms, it means that the expected gain from increased investment more than compensated for the expected costs of financial distress. Thus one needs to understand how, and under what conditions, this private calculation leads to inefficient decisions at the social level. What is the externality? Likewise, how can it be remedied? Do we need regulation and government intervention, or can innovative market structure that internalizes the externality solve the problem.

There is a literature in the wake of the U.S. financial crisis that has focused on pecuniary externalities as the source of the problem. This literature seeks policy interventions and regulations to remedy the associated distortions, e.g., balance sheet effects, amplifiers and fire sales. Under pecuniary externalities, trading on a market adversely affects others via the revaluation of traded items. Solutions range from regulation of portfolios, restrictions on saving or credit, interest rate restrictions, fiscal policy, or taxes and subsidies levied by the government. However, general equilibrium theory suggests in other contexts that bundling, exclusivity and suitably designed additional markets for the objects associated with externalities could internalize those externalities, without the need of further policy interventions, or the need to quantify interventions, as the latter requires yet more information.[1]

The influence of prices which can cause inefficiencies is akin to pollution, which has a remedy in competitive markets for the rights to pollute. We will draw an analogy between pollution and price externalities to explain what we do. Specifically, for pollution, consider an initial economy with two goods, one period, one representative price-taking consumer and one representative price-taking firm. The consumer is endowed with one good which can be consumed or used by firms to produce the second good which the household also values.

[1]To reiterate, our solution to the problem of pecuniary externalities is a market-based solution, creating market infrastructure, rights to trade, and prices of those rights in such a way that a constrained-efficient allocations can be decentralized with a price system. The two fundamental welfare theorems hold. There is no need for the government to impose quotas, allocate rights or calculate marginal taxes, as arguably these require considerable information and nontrivial calculation. It is beyond the scope of this paper to repeat the arguments of Smith and Hayek, the Lange Lerner debates, and the work of Hurwicz, for example, but we do come back to the determination of prices in a section with the planner transformed into market maker and in the conclusion, with citations to the computation of equilibria and explicit market making mechanisms.

However, that production comes with air or water pollution, which gives the household disutility. The competitive equilibrium in which this pollution is not priced is not at a social optimum; marginal rates of substitution in consumption and production do not line up, as they would in the planner's problem. But now suppose we create markets in the rights to pollute. Factories have to buy rights to emit pollution, a cost which lowers their profit. They choose how much to produce and how much to pollute, consistent with permits purchased. Households sell rights to suffer pollution, a revenue added to their budget, and choose how much pollution they want and how much to consume of the two goods. In the new decentralized market equilibrium, the demand for rights to pollute by firms and the supply of rights to suffer pollution by households will be equated by the appropriate price of rights, and money in equal amounts changes hands. The new equilibrium is Pareto optimal; it has some but less pollution. Of course, there needs to be some enforcement.[2]

Now consider an analogue economy with two goods, two periods, and two representative price-taking households, types a and b. There is no uncertainty. The decisions are intertemporal decisions, over the two time periods, and within-period decisions, across the two goods. Further, suppose that if there were no obstacles to trade and if markets were complete, then the environment is such that, in a competitive equilibrium, type a would be a lender and type b a borrower. However, suppose in contrast that borrowing cannot happen in equilibrium.[3] Further, only one of the two goods can be stored, good z. The relatively rich type a ends up smoothing consumption over time on its own, not by lending to type b but by saving good z. As a result, the price of the storage good z is low in the second period, as type a sells good z in the spot market then. This relative price is moving with saving, but both types take this equilibrium price as given. This relative price is the source of a pecuniary externality.

The stored good in this intertemporal example is like the input good in the initial pollution example. As with the pollution example where we needed markets for rights to pollute, here with this pecuniary externality we need markets for rights to trade at the future relative

[2]Firms cannot pollute beyond rights purchased, as in cap and trade. The difference between cap and trade and the full market solution described here is that the quantity of permits is market determined and not fixed by the government.

[3]This can be derived from limited commitment, that the type b borrowers' promise to pay must be backed by collateral held in storage. By assumption the would-be borrower type b has little of the collateral good, and so the competitive equilibrium with limited commitment has no borrowing and lending.

price. Agents of each type choose in the first period the relative price at which they want to trade in the second period. That is, agents choose in the first period one price from among various possible prices they want to prevail in the future. One can think of a first period exchange or trading house earmarked by each possible such relative price, p. Subsequently, we refer to these first period exchanges as price exchanges or for brevity, p-exchanges. Each agent can choose the price exchange it wants without regard to what any other agent is doing. But crucially, the right to trade in each and every p-exchange is priced. The fee structure has a per unit price and quantity decomposition, a price times future type-specific excess demand, as determined by the relative price p of the price exchange chosen and previously chosen saving. The fees to engage and trade rights in these price exchanges are the market-determined decentralizing feature.

Our analysis extends well beyond the example, which is intended to be illustrative. In our more general setup any agent type can make a promise to deliver in the future, but all promises must be backed by sufficient collateral so that promises can be honored. We can allow uncertainty about future states of the world, and promises can be state-contingent, as would be the collateral constraints, holding state by state to ensure state-contingent promises are honored. We can also allow exogenously incomplete markets. With incomplete security markets we can drop the collateral requirement, but generically competitive equilibria are inefficient due to pecuniary externalities when there are multiple goods in spot markets. Trades in securities markets determine the distribution of income in spot markets, but by definition, when markets are incomplete, there is no way to hedge the resulting income movements across states (e.g., Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986). In this case, as security positions move relative prices in multiple future states, rights are naturally a vector of rights over future states. We can remedy the pecuniary externality, so that allocations are constrained-efficient, though still not complete. Our solution is not about completing markets but remedying price externalities. [4] Our solution works in general

[4]Other environments include a fire sale economy (Lorenzoni, 2008) as per our introduction to this paper, and a liquidity-constrained economy (Hart and Zingales, 2013), where there is too much saving. We also extend our method to environments with information imperfections, namely a moral hazard contract economy with multiple goods and retrade in spot markets (e.g., Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011) and a Diamond-Dybvig economy where an agent's excess demands in interim bond markets is not known ex ante as each agent is subject to unobserved preference shocks that determine the direction of trade (e.g., Diamond and Dybvig, 1983; Jacklin, 1987). These environments are formally described in Online

as price is a "sufficient statistic" for the source of the problem, regardless of the underlying environment. This is after all the nature of pecuniary externalities. For more details see Online Appendix H.

Our contribution is related to Coase (1960) in its emphasis on rights. Our pollution example is one of his lead examples. However, the Coase theorem is about how any given initial arbitrary distribution of rights would not matter if there were bargaining and no trading frictions, just as the initial allocation of rights to pollute in cap and trade would not matter, as efficiency works through opportunity costs.[5] In contrast, for us, rights are market determined. Thus, we are working not in the tradition of Coase (1960) but rather in the tradition of Arrow (1969), following Meade (1952), on the equivalence of solutions to planning problems and competitive equilibria with rights to trade in the objects causing non-pecuniary externalities. Keys are additional markets and excludability.[6] Of course we focus on pecuniary externalities. Finally, a point of emphasis, a comparison between what we do for pecuniary externalities and what Arrow does for consumption externalities. For Arrow, rights to specify consumption for others are priced and traded, with decentralized utility maximization problems and market clearing. Here we do not have such direct rights on consumption of others, but the solution takes into account that others' consumptions are impacted through their excess demands by the market price chosen. We have a market for that market price. Otherwise, we have, as in Arrow, decentralized utility maximization problems and market clearing in rights.[7]

The remainder of the paper proceeds as follows. Section 2 presents the saving economy to illustrate the ingredients. This includes Section 2.1 with complete markets which achieves first-best Pareto optimality. Section 2.2 describes a competitive economy with incomplete

Appendix H.

[5]We can relate our solution to Lindahl (1958) who uses agent-specific prices to solve a public goods problem when there is heterogeneity in willingness to pay. Though the per unit price of an exchange is common, type specific excess demands make the total fee agent specific.

[6]See Chapter 11 of Mas-Colell et al. (1995) for more about this distinction between Coase (1960) and Arrow (1969). Interestingly, Arrow (1969) is less concerned about excludability, an intrinsic part of creating the necessary markets, as he feels this has a natural counterpart in many real world problems. Arrow (1969) is more concerned about the obvious small numbers problem. However, this part is easy to remedy. The theory applies to competitive markets (e.g., financial markets) where agents are price takers but groups of agents (agent types) can create a pecuniary externality.

[7]See also consumption rights in Bisin and Gottardi (2006) who deal with adverse selection.

markets generating a pecuniary externality. Section 2.3 presents the basic planner problem, making clear that the planner can take into account the equilibrium pricing function in the second period spot markets, the mapping from savings to price and how this impacts agent type value functions. Section 2.4 presents an equivalent market maker problem, introducing the language of rights and letting price be the planner control variable, as this provides a transition to the decentralized markets with trading rights in Section 2.5. A general economy is described and the welfare theorems and existence theorem are stated in Section 3. Section 4 concludes with some comments on implementation. The Appendix in this paper presents the proof of the second welfare theorem, and additional results are in the Online Appendices.

2 A Saving Economy Illustrative of the Key Ingredients

This section features in notation the example economy of the introduction, a saving economy with no uncertainty. There are two periods, $t = 1, 2$. Planning takes place at the initial date, hence $t = 1$, though there are spot markets and saving later in that period, as well. The second date is $t = 2$, and there is no uncertainty. This is a pure intertemporal economy, making the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades in financial options.

There is a continuum of agents of measure one. The agents are however divided into two heterogeneous types $h = a, b$. Each type h consists of α^h fraction of the population. There are two consumption goods, which can be traded and consumed in each period t. Each unit of good z stored will become R units of good z at date $t = 2$. Good w cannot be stored (is completely perishable). Let $k^h \in \mathbb{R}_+$ denote the saving (equivalent to the holding of good z) of an agent type h at the end of period $t = 1$ to be carried to period $t = 2$. The contemporary preferences of agent type h are represented by the utility function $u^h\left(c_w^h, c_z^h\right)$, which is continuous, strictly concave, strictly increasing in both consumption goods, and satisfies the usual Inada conditions. Each agent type h is endowed with good w and good z, $\mathbf{e}_t^h = \left(e_{wt}^h, e_{zt}^h\right) \in \mathbb{R}_+^2$ in period $t = 1, 2$.

For the numerical example we shall suppose each of the two types has an identical constant relative risk aversion (CRRA) utility function $u^h(c_w, c_z) = -\frac{1}{c_w} - \frac{1}{c_z}$, $h = a, b$. The endowment profiles are such that an agent type a is well endowed with 3 units of both goods in period $t = 1$ relative to one unit of both at $t = 2$, a savings type, and vice versa for type b, a want-to-be-borrowing type. Each type h consists of $\frac{1}{2}$ fraction of the population, i.e., $\alpha^h = \frac{1}{2}$. Finally, set $R = 1$.

2.1 Review of the Basics: First Best Planner Problem and Complete Markets

The basic planner problem that delivers first-best unconstrained allocations without externalities, and the associated dynamic complete markets implementation, are each reviewed here in this subsection. This then sets the stage in the next subsection for the introduction of the externality generated by incomplete markets and for a comparison with a suitably modified planner problem that delivers constrained-optimal allocations, distinct from those with incomplete markets.

In the basic problem, the planner will maximize a λ-weighted sum of discounted utilities subject to constraints: the non-negativity constraint on saving $k^h \geq 0$; the resource constraint in the first period for good w, that aggregated consumption c^h_{w1} cannot exceed the aggregate endowment; the resource constraint in the first period for good z, that aggregated consumption c^h_{z1} cannot exceed the aggregate endowment subtracting aggregate saving; and the resource constraints for the second period, that transfers $\tau^h_{\ell 2}$ of each good w and z sum to zero.

Definition 1 (Basic Planner Problem)**.**

$$\max_{\left(c^h_{w1}, c^h_{z1}, k^h, \tau^h_{w2}, \tau^h_{z2}\right)_h} \sum_h \lambda^h \alpha^h \left[u^h\left(c^h_{w1}, c^h_{z1}\right) + u^h\left(e^h_{w2} + \tau^h_{w2}, e^h_{z2} + Rk^h + \tau^h_{z2}\right)\right] \tag{1}$$

subject to

$$k^h \geq 0, \forall h, \tag{2}$$

$$\sum_h \alpha^h c^h_{w1} = \sum_h \alpha^h e^h_{w1}, \tag{3}$$

$$\sum_h \alpha^h \left[c^h_{z1} + k^h\right] = \sum_h \alpha^h e^h_{z1}, \tag{4}$$

$$\sum_h \alpha^h \tau^h_{\ell 2} = 0, \forall \ell = w, z. \tag{5}$$

Note that under the usual regularity assumptions, this program is concave and non-negativity constraints on consumption can be ignored. The necessary and sufficient first-order conditions for all these first-best optima are that marginal rates of substitution are equated across agents a and b at each date, and intertemporal Euler equations hold with possible inequality if the non-negativity constraint on saving $k^h \geq 0$ is binding or adjusted by a Lagrange multiplier, respectively.

$$\frac{u_{z1}^a}{u_{w1}^a} = \frac{u_{z1}^b}{u_{w1}^b}, \frac{u_{z2}^a}{u_{w2}^a} = \frac{u_{z2}^b}{u_{w2}^b}, \tag{6}$$

$$u_{z1}^h = R u_{z2}^h + \frac{\mu^h}{\lambda^h \alpha^h}, \forall h = a, b, \tag{7}$$

where $u_{\ell t}^h \equiv \frac{\partial u^h\left(c_{wt}^h, c_{zt}^h\right)}{\partial c_{\ell t}}$ for $\ell = w, z; t = 1, 2$, and μ^h is a Lagrange multiplier associated with $k^h \geq 0$.

There is an entire class of first best allocations as solutions to the planner problem indexed by λ-weights, which pin down levels.

$$\lambda^h \alpha^h u_{\ell t}^h = \mu_{\ell t}, \forall h = a, b; \ell = w, z; t = 1, 2, \tag{8}$$

where $\mu_{\ell t}$ are Lagrange multipliers on the resource constraints for goods $\ell = w, z$ at $t = 1, 2$.

These first-best optimal allocations can be achieved in complete markets at the planning date $t = 0$. However, anticipating the discussion which follows, we instead adopt the standard dynamic implementation with trade in spot markets at $t = 2$, then moving back to spot markets at $t = 1$ along with financial securities θ^h, $h = a, b$, purchased (or sold) at $t = 1$ and paying a unit good z at $t = 2$. As its part of the underlying environment, we retain the possibility of physical savings k^h. Henceforth, let good w be the numeraire good at each date t with the relative price of good z as p_t, $t = 1, 2$, and relative price of the financial security as Q.

Definition 2. A competitive equilibrium with complete markets is a specification of prices p_t of good z in period $t = 1, 2$, and the price of financial security Q at $t = 1$; consumptions $\left(c_{w1}^h, c_{z1}^h\right)$ at $t = 1$, saving and financial securities $\left(k^h, \theta^h\right)$ decisions made at $t = 1$, and trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ at $t = 2$ for each type $h = a, b$ such that (i) at $t = 2$, taking $\left(p_2, k^h, \theta^h\right)$ as given parameters at $t = 2$, agent type $h = a, b$ solves

$$V^h\left(k^h, \theta^h, p_2\right) = \max_{\tau_{w2}^h, \tau_{z2}^h} u^h\left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + R k^h + \theta^h + \tau_{z2}^h\right) \tag{9}$$

subject to the budget constraint in period $t = 2$,

$$\tau_{w2}^h + p_2 \tau_{z2}^h = 0, \tag{10}$$

(ii) at $t = 1$, taking (p_1, Q) and $V^h\left(k^h, \theta^h, p_2\right)$ from $t = 2$ as given, agent type $h = a, b$ solves

$$\max_{c_{w1}^h, c_{z1}^h, k^h, \theta^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + V^h\left(k^h, \theta^h, p_2\right) \tag{11}$$

subject to the non-negative saving constraint (2), and the budget constraint in period $t = 1$,

$$c_{w1}^h + p_1\left(c_{z1}^h + k^h\right) + Q\theta^h = e_{w1}^h + p_1 e_{z1}^h, \tag{12}$$

(iii) market clearing conditions hold, specifically, the resource constraints in the planner problem: for good w, (3); good z at $t = 1$, (4); trades at $t = 2$, (5); and financial securities at $t = 1$

$$\sum_h \alpha^h \theta^h = 0. \tag{13}$$

One can find a complete markets competitive equilibrium using the shadow prices from the basic planner problem:[8]

$$p_1 = \frac{\mu_{z1}}{\mu_{w1}}, p_2 = \frac{\mu_{z2}}{\mu_{w2}}, Q = \frac{\mu_{z2}}{\mu_{w1}}. \tag{14}$$

The first-best competitive equilibrium allocation for the numerical example is displayed in Table 2. It features no physical savings, $k^a = k^b = 0$, and constant prices $p_1 = p_2 = 1$. Each agent consumes 2 units of each good, w and z, at each period $t = 1, 2$. There is an active financial lending/borrowing market; agents type b borrow 1 unit of good z at $t = 1$ with repayment at $t = 2$ and agents type a are on the other side of this market, lending 1 unit at $t = 1$ with returns at $t = 2$. Finally, price $Q = 1$. The Lagrange multipliers on physical savings constraints are zero. There is no need to move good z around intertemporally.

[8]The basics, worthy of review at this point, there is a difference between the planner problem and the decentralization: the planner is taking into account each and every resource constraint, which is how the shadow prices are generated, as Lagrange multipliers. The agents simply take these prices as given for all time and maximize utility subject to budget constraints, ignoring what everyone else is doing. To anticipate, exactly the same steps will be used below to decentralize a constrained planner problem when there is an externality generated by prices, with a shadow price coming from the constraint that as an equilibrium condition excess demands at $t = 2$ sum to zero.

Table 1: Equilibrium allocations with externalities.

	k^h	c^h_{w1}	c^h_{z1}	c^h_{w2}	c^h_{z2}	$U^h\left(\mathbf{c}^h\right)$
$h=a$	0	2	2	2	2	-2.00
$h=b$	0	2	2	2	2	-2.00

2.2 Competitive Equilibrium with Incomplete Markets, Saving Only

Here we eliminate the financial securities market, which can be taken as an exogenous restriction or something derived as endogenous when there are collateral constraints on borrowing as in footnote 3. The notation of the complete markets definition survives almost in tact except for the disappearance of θ^h and Q.

Definition 3. A competitive equilibrium with incomplete markets, specifically saving k^h only and no securities, is a specification of prices p_t of good z in period $t=1,2$; consumptions $\left(c^h_{w1}, c^h_{z1}\right)$ at $t=1$, saving k^h decision made at $t=1$, and trades $\left(\tau^h_{w2}, \tau^h_{z2}\right)$ at $t=2$ for each type $h=a,b$ such that (i) at $t=2$, taking $\left(p_2, k^h\right)$ as given parameters, agent type $h=a,b$ solves for trades $\left(\tau^h_{w2}, \tau^h_{z2}\right)$ subject to budget (10) generating the value function $V^h\left(k^h, p_2\right)$, (ii) at $t=1$, taking p_1 as given, agent type $h=a,b$ maximizes (11) using the derived $V^h\left(k^h, p_2\right)$ from (i) subject to the non-negative saving constraint (2), and the budget constraint in period $t=1$, equation (12), with $\theta^h=0$, (iii) clearing the markets for good w, (3); and good z at $t=1$, (4); and trades at $t=2$, (5), respectively, while dropping securities clearing condition (13).

Necessary conditions for competitive equilibrium related to saving k^h are

$$p_1 = \frac{u^h_{z1}}{u^h_{w1}} = \frac{u^h_{z2}}{u^h_{w1}} R + \frac{\eta^h}{\lambda^h \alpha^h u^h_{w1}}, \forall h=a,b. \tag{15}$$

where $u^h_{\ell t} \equiv \frac{\partial u^h\left(c^h_{wt}, c^h_{zt}\right)}{\partial c_{\ell t}}$ for $\ell = w, z; t=1,2$, and η^h is the Lagrange multiplier for the non-negative saving constraint (2) for an agent type h.

To reiterate, the excess demands for good w at $t=2$, the maximizing choice $\tau^{h*}_{w2}\left(k^h, p_2\right)$ in (9) over types h, must satisfy the market-clearing conditions for trades (5) at $t=2$ (with no θ^h, of course).[9] In fact, the spot market equilibrium price p_2 of Definition 3 can be defined

[9]With Walras' law, excess demands for good z are implied.

as the price which makes (5) be satisfied, and this is a function of savings of both types, possibly zero, predetermined at $t = 2$, here keeping endowments implicit: $p_2 = p_2\left(k^a, k^b\right)$. In fact, this pricing function applies at $t = 2$ for all possible specifications of agent savings and can be thought of as an equilibrium consistency constraint for prices and savings, with prices not only for actual savings along the equilibrium path but also any counter-factual savings as if different savings decisions had been made. Of course, agents take the equilibrium path of prices as given, not this function.

Table 2: Equilibrium allocations with externalities.

	k^h	c^h_{w1}	c^h_{z1}	c^h_{w2}	c^h_{z2}	$U^h\left(\mathbf{c}^h\right)$
$h = a$	0	2	2	2	2	-2.00
$h = b$	0	2	2	2	2	-2.00

For the numerical example, we summarize the equilibrium allocation in Table **??** featuring physical saving k^h and consumption $c^h_{\ell t}$. The incomplete markets equilibrium will have agent type b borrowing nothing as it is not allowed and only trading in spot markets. Agent type a will be holding physical saving on its own to smooth consumption over time. To iterate, $k^b = 0$ and $k^a > 0$. The price of good z in period $t = 1$ is $p^{ex}_1 = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and at date 2 is $p^{ex}_2 = 0.5570$. Note that the price of good z is high at $t = 1$ relative to the first best since relatively much is put into storage, and likewise the price of good z is low at $t = 2$ when the storage is sold on the market.

2.3 The Planner Problem for Constrained-Efficient Allocations

The planner maximizes λ-weighted sum of agent type utilities by choice of the distribution of savings and first-period consumptions to solve the following programming problem:

$$\max_{\left(c^h_{w1}, c^h_{z1}, k^h\right)_h} \sum_h \lambda^h \alpha^h \left[u^h\left(c^h_{w1}, c^h_{z1}\right) + V^h\left(k^h, p_2\left(k^a, k^b\right)\right)\right] \tag{16}$$

subject to non-negative constraints on saving (2), and the resource constraints for good w and good z at $t = 1$, (3) and (4), respectively. Note that the value function V^h is already defined in the agent maximization problem of Definition 3 with the pricing function $p_2\left(k^a, k^b\right)$ inserted.

The planner must respect that allocations will be determined in spot markets at $t = 2$ with endogenous market clearing prices $p_2 = p_2\left(k^a, k^b\right)$. But the planner is aware that savings influence the relative price, that agents in contrast do not take into account.[10] Constrained-optimal allocations take into account two key features. First, the planner cannot have more objects under her control than the agents do, i.e. the planner cannot impose a solution with borrowing at $t = 1$ to be repaid at $t = 2$ (that is, the planner cannot undo the incomplete markets). The planner can assign $t = 1$ consumption and savings, as even with incomplete markets agents do choose these. Second, in contrast, the planner cannot assign consumptions at $t = 2$. An agent type h with saving k^h is free to choose even in the implementation of the planner problem spot trades $\left(\tau^h_{w2}, \tau^h_{z2}\right)$ to solve utility maximization of Definition 3, yielding, as noted earlier, the value function $V^h\left(k^h, p_2\right)$. But the planner does take into account that the equilibrium relative price p_2 will be determined by assigned savings at $t = 1$, and can try out various possible savings.

The necessary conditions for constrained optimality are given by

$$\frac{u^h_{z1}}{u^h_{w1}} = \frac{u^h_{z2}}{u^h_{w1}} R + \frac{\mu^h}{\lambda^h \alpha^h u^h_{w1}} + \frac{1}{\lambda^h u^h_{w1}} \sum_{\tilde{h}} \lambda^{\tilde{h}} \alpha^{\tilde{h}} \frac{\partial V^{\tilde{h}}}{\partial p_2} \frac{\partial p_2}{\partial k^h}, \forall h = a, b. \tag{17}$$

where μ^h are Lagrange multipliers for the non-negative saving constraints. The first two components were there also in the competitive equilibrium equation (15), but the third one here is new. It is the effect on the future $t = 2$ price from the impact of savings through the equilibrium price function, $\frac{\partial p_2\left(k^1, k^2\right)}{\partial k^{\tilde{h}}}$ for all agent types, something which is quite naturally taken into account by the planner.[11]

This is also the correct point to review the nature of the problem of pecuniary externalities. First, we summarize the discussion thus far:

Definition 4. Distinction between Constrained-Optimal and Incomplete Markets Allocations, when borrowing is not allowed: The solutions to the Planner Problem (16) are termed constrained-efficient allocations. When (17) is different from (15), the competitive equilibrium with no borrowing is not constrained-efficient.

[10] This is the source of pecuniary externality in the decentralized competitive equilibrium as defined formally below and the analogy to pollution as in the introduction.

[11] Given that the constraint set is not convex, the optimality conditions are necessary but may not be sufficient. This does not cause any problem to the argument here, as we simply need to show conditions under which an equilibrium cannot be constrained optimal, i.e., does not satisfy the necessary optimal conditions (17). We overcome the non-convexity problem using a mixture representation as in the Appendix, where first-order conditions are both necessary and sufficient.

It is sufficient for this divergence that the last term in (17) be nonzero.[12]

Definition 5. Pecuniary Externality: A pecuniary externality arises when the last term in (17) is non zero.

For the numerical example, the solution to the programming problem (16) are presented in Table 3 below.[13]

2.4 Transition to the Decentralization: Market Maker Problem

Guided by the literature, a way to eliminate externalities is to create a market in the object causing the externality, e.g., markets for rights to pollute. As the price is the source of the pecuniary externality, this suggests that somehow agents here should be choosing prices and buying rights to trade at those prices. Here we provide a transition to that decentralization, reinterpreting the planner as choosing prices and assigning rights to trade at those prices. More formally, we can transform the problem of the planner to a fully equivalent one in which the planner is a market maker choosing price p_2 and rights to trade at that price, and then finding savings $\boldsymbol{k}$ consistent with the pricing function $p_2\left(k^a, k^b\right)$ to support that price p_2 with all possible prices considered. We do this in a series of steps.

First step, as an alternative to using the direct spot equilibrium price map $p_2\left(k^a, k^b\right)$ yielding p_2 for any choice of k^a and k^b, one can work with the inverse equilibrium price map, that is, designating p_2 first and then filling in the requisite k^h, $h = a, b$. For example, as with the example economy with homogeneous homothetic preferences, $R = 1$, and agent type a endogenously doing all the savings, given any price $p_2 = p_2\left(k^a, k^b = 0\right) \equiv g\left(k^a\right)$, we can use

[12] Even in the first best, an expression similar to (17) appears, but due to the dynamic decentralization, the last term is zero. Some intuition: In the first-best the λ-weighted derivative of agent type h value function with respect to k^h are equated, across agent types. What generates a pecuniary externality in the competitive decentralization are genuinely incomplete security markets with multiple goods and saving, coupled with trading in spot market, as in Geanakoplos and Polemarchakis (1986). Indeed, the example of Greenwald and Stiglitz (1986) is close to our baseline savings-only example here; we establish in Online Appendix G that the competitive decentralization (without rights) is not achieving a constrained optimum, as prices are moving with saving, and in Kilenthong and Townsend (2014) that there is too much saving (compare the higher number in Table **??** with Table 3).

[13] Note that, with the pecuniary externality the shadow price η^h on $k^h \geq 0$ in (15) is different from the μ^h in (17) as well, a natural fall-out from changes in the physical savings k^a between the two solutions.

the inverse function $g^{-1}(p_2)$ taking $\boldsymbol{e}_2^h$ as given to go from price p_2 to the requisite savings k^a,

$$k^a = \frac{\sum_h \alpha^h e_{w2}^h}{g^{-1}(p_2)} - \sum_h \alpha^h e_{z2}^h. \tag{18}$$

But equation (18) is only intended as an illustrative example of the inverse equilibrium price map. More generally, both agents could in principle be saving, and preferences need not be homothetic. Further, equilibrium prices need not be unique, though they would be under gross substitutes. For the direct map, under a given k^h, $h = a, b$, there may be multiple equilibria p_2. We only require that there be an equilibrium. For the inverse map for a given p_2, there may be multiple consistent k^h, $h = a, b$, and in that case some selection would be made. We can then state the Constrained Planner problem (16) as an equivalent Market Maker problem, namely choosing c_{w1}^h, c_{z1}^h, and now choosing p_2 as a control subject to the inverse equilibrium price map, the date $t = 1$ resource constraints, and non-negative physical savings constraints. A different choice of p_2 would entail an appropriate adjustment of k^h, $h = a, b$. In effect, p_2 becomes a parameter for the k^h. In sum, for any choice of p_2 considered by the planner, the spot market clearing condition at $t = 2$ is satisfied. This first step is meant to clarify the conceptualization of the problem and the decision to choose prices as a control. In fact, in the second step, we dispense with the mapping language and define equilibrium prices implicitly through market clearing of excess demands.

Second step, define $\Delta^h\left(k^h(p_2), p_2\right) \equiv \tau_{w2}^{h*}\left(k^h(p_2), p_2\right)$ where again the right-hand side is the type h maximizing choice. We can then replace the inverse equilibrium price map by simply rewriting clearing condition (5) for good $\ell = w$ as

$$\sum_h \alpha^h \Delta^h\left(k^h(p_2), p_2\right) = 0, \forall p_2. \tag{19}$$

Constraints (19) define p_2 implicitly and hold for any possible choices of p_2, as a predetermined policy parameter, just as the direct function $p_2 = p_2\left(k^a, k^b\right)$ held for any actual and counterfactual choice of the k^h's as predetermined saving. Crucially, market clearing constraint (19) picks up for any given p_2 a Lagrange multiplier in the constrained planner sub-problem when p_2 is the control, denoted $\mu_\Delta(p_2)$, for the given p_2. We do this for all possible p_2.

Third step, notationally, write out the entire vector of variables for any agent h given the planner's choice of p_2. As with saving k^h as a function of p_2, that is $k^h(p_2)$, all control variables at $t = 1$ are written given the policy choice p_2 namely, $\mathbf{x}^h(p_2) = \left[\mathbf{c}_1^h(p_2),\ k^h(p_2),\ \Delta^h\left(k^h(p_2), p_2\right)\right]$. Likewise, from $k^h(p_2)$ the value function V^h is now here rewritten as function of p_2, as in (20) below.[14]

Fourth step, and anticipating the decentralization in Section 2.5 below where each agent chooses price p_2 for spot trading, possibly distinct across types, we let the constrained planner choose the fraction[15] $\delta^h(p_2)$ of type h assigned to p_2-exchange varying over all possible prices p_2.[16]

Thus the constrained planner problem becomes the equivalent Market Maker problem defined as

$$\max_{\left[\delta^h(p_2), \mathbf{x}^h(p_2)\right]_{h,p_2}} \sum_h \sum_{p_2} \lambda^h \alpha^h \delta^h(p_2) \left[u^h\left(c_{w1}^h(p_2), c_{z1}^h(p_2)\right) + V^h\left(k^h(p_2), p_2\right)\right] \tag{20}$$

subject to non-negative constraints on saving, the resource constraints for goods w and z at $t = 1$, and the spot-market or consistency constraints (24),

$$\delta^h(p_2) k^h(p_2) \geq 0, \forall h; p_2, \tag{21}$$

$$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) c_{w1}^h(p_2) = \sum_h \alpha^h e_{w1}^h, \tag{22}$$

$$\sum_{p_2} \sum_h \alpha^h \delta^h(p_2) \left[c_{z1}^h(p_2) + k^h(p_2)\right] = \sum_h \alpha^h e_{z1}^h, \tag{23}$$

[14]One can think of the planner as choosing all controls, price and the quantities for consumption, saving, and excess demand simultaneously, in which case it is redundant to index the quantities by price p_2. However, we wish to emphasize the planner as a market maker choosing p_2 and all the other objects which need to be aligned with that choice of p_2. The other objects will be different if p_2 were different. This also sets the stage for the decentralization, next.

[15]We need fraction $\delta^h(p_2)$ of type h assigned to p_2-exchange varying over all possible prices p_2 to derive condition (26) below. See Online Appendix A. Also the fractions $\delta^h(p_2)$ of type h assigned to p_2-exchange as controls help to ensure existence of a decentralized equilibrium, generating sufficient assumptions, concavity of relevant functions and convexity of choice sets. See the discussion of the existence of equilibrium in the Appendix. Typically, the solution will set the fractions to be zero or unity, and $\delta^h(p_2)$ to be equal across types, a common choice of p_2-exchanges, but we need not impose this as a constraint.

[16]As written the solutions to the planner problem (16) are a subset of the solutions to market maker problem (20), defined below. Though we deliberately kept (16) as simple and clear as possible, any solution to (20) which has active groups can be reversed engineered back from the indirect map from p_2 to k, so if the assignments for p_2 are different, so are the assignments for k. In the case of multiple k^h consistent with p_2, we simply choose the one with the same allocation.

$$\sum_h \alpha^h \delta^h (p_2) \Delta^h \left(k^h (p_2), p_2\right) = 0, \forall p_2. \tag{24}$$

The necessary conditions for constrained optimality and its associated Lagrange multipliers can be derived in two parts. First, for each h with fraction $\delta^h (p_2) > 0$, for a given arbitrary choice of p_2, the planner is choosing $c_{w1}^h (p_2)$, $c_{z1}^h (p_2)$ and $k^h (p_2)$ to maximize (20) subject to (21)-(24):

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h} R + \frac{\mu^h (p_2)}{\lambda^h \alpha^h u_{w1}^h} - \frac{\mu_\Delta (p_2)}{\mu_{w1}} \Delta_k^h \left(k^h (p_2), p_2\right), \forall h = a, b, \tag{25}$$

where the derivative of $\Delta_k^h \left(k^h, p_2\right) \equiv \frac{\partial \Delta^h \left(k^h, p_2\right)}{\partial k}$; $\mu_{\ell t}$ are Lagrange multipliers for the resource constraints for good $\ell = w, z$ in period $t = 1, 2$; $\mu^h (p_2)$ are Lagrange multipliers for the non-negative saving constraints; and $\mu_\Delta (p_2)$ is the key Lagrange multiplier for the consistency constraints (24) at the arbitrarily chosen p_2. We thus get a shadow price for rights for any p_2 including those eventually not chosen in the global solution. See Online Appendix A for more details. This is a familiar Euler equation with marginal utility at $t = 1$, adjusted by the potentially binding $k^h \geq 0$, equal to the marginal utility at $t = 2$ times rate of returns R, adjusted for the binding consistency constraint (24) at p_2. In the decentralized version, Section 2.5, the shadow prices for goods and for the consistency constraint (24) at each p_2 become market prices.

Second, regarding the global problem, the overall choice of p_2, for each h with fraction $\delta^h (p_2) > 0$, satisfies the following condition (see Online Appendix A for the derivation):

$$\lambda^h V_p^h \left(k^h (p_2), p_2\right) = \mu_\Delta (p_2) \Delta_p^h \left(k^h (p_2), p_2\right) \tag{26}$$

The planner trades off the marginal benefit (cost) on V^h from choosing p_2 with the marginal cost (benefit) on excess demand from choosing p_2. Note that the shadow prices on rights derived earlier for arbitrary p_2 are being used here, to find the optimizing p_2, the first order condition for which is (26). Note also that λ^h will be proportional to the inverse shadow price on consumer h budget constraint in the decentralized version, Section 2.5.

The numerical example is displayed in Table 3 where, for the constrained-optimal allocation, $p_2^{op} = 0.5974$ is higher than in the competitive equilibrium with externalities in Table **??** as there is less saving carried over to $t = 2$.[17] Note also that $\Delta^h \left(p_2^{op}\right) = 0.30$ for $h = a$ and -0.30 for $h = b$.

[17]The Pareto weights are chosen so that the optimal allocation corresponds to the competitive equilibrium with rights to trade, defined below in Section 2.5, without transfers.

Table 3: Constrained Optimal Allocation with Pareto weights $\lambda^1 = 0.778$ and $\lambda^2 = 0.222$.

	k^h	c^h_{w1}	c^h_{z1}	c^h_{w2}	c^h_{z2}	$\Delta^h\left(p_2^{op}\right)$	$U^h\left(\mathbf{c}^h\right)$
$h = a$	1.18	2.61	1.84	1.30	1.68	0.30	-2.2934
$h = b$	0	1.39	0.98	2.70	3.50	-0.30	-2.3904

Finally, and in transition to the next section, one could refer to the $\Delta^h\left(k^h\left(p_2\right), p_2\right)$ as rights. The planner as market maker assigns these rights for spot market trades at $t = 2$ at price p_2, but these rights must be consistent with the trades agent type h would want to do voluntarily at that time and price. The rights notation appears as redundant at this point, but as noted, this is an important step in the reformulation of the planner problem to the market markers problem as a transition to the decentralization. That is, in the decentralized problem, agents choose the rights, so we let the planner as market maker choose these too. Of course, agents in the decentralized problem like the market maker here will choose p_2 as well.

2.5 Decentralization with Individually Chosen Rights to Trade Δ^h at prices P_Δ

It is now a straightforward step to decentralize the market maker problem in the previous section. Each agent will maximize utility subject to budget constraints taking all prices as given. Of course, no market clearing constraints are included in this utility maximization problem. But prices must be such that all markets clear, that is, the resource constraints must be satisfied.

Let the type h choice of exchange p_2 be described by indicators $\delta^h\left(p_2\right)$ taking on a value zero or one. That is, p_2 is a discrete choice with $\delta^h\left(p_2\right) \geq 0$ for all p_2 and $\sum_{p_2} \delta^h\left(p_2\right) = 1$. Only one p_2 can be positive under $\delta^h\left(p_2\right)$, the one for the chosen p_2.[18] The rest of type h choices are indexed by it. This $\delta^h\left(p_2\right)$ can in principle be different for different agents. Choices are made independently. Type h commodity point also includes consumption, saving and rights, that is, the choice is $\left(\mathbf{x}^h\left(p_2\right), \delta^h\left(p_2\right)\right)_{p_2}$, where $\mathbf{x}^h\left(p_2\right) \equiv \left[\mathbf{c}^h_1\left(p_2\right), k^h\left(p_2\right), \Delta^h\left(k^h\left(p_2\right), p_2\right)\right]$. The type-specific excess demand Δ^h as function of p_2 and k^h is known and given but depends on the choice of k^h, which is of course endogenous.

[18]One can also think of $\delta^h\left(p_2\right)$ as the probability type h is assigned to p_2, so what is taken as a fraction by the planner becomes a probability for the agent.

Note that Δ^h can be negative if type h is supplying. These $\Delta^h\left(k^h(p_2), p_2\right)$ play the role of rights to enter designated p_2-exchanges at $t = 2$.

For the decentralization, as is standard, simply replace the shadow prices $\mu_\Delta(p_2)$ in the market maker problem with the actual price $P_\Delta(p_2)$. These will be equivalent. More generally, for market prices, let p_1 denote the price of good z in period $t = 1$, p_2 denote the price of good z in period $t = 2$, and $P_\Delta(p_2)$ denote the key price of rights to trade in spot markets at $t = 2$. Again, all these prices are taken as given by individual agents in their maximization problems.

Trades are sequential over time. The initial endowments e^h_{w1} and e^h_{z1} at $t = 1$ are sold at consumption prices 1 and p_1 for goods w and z, respectively, regardless of the p_2-exchange chosen. Then, in the chosen p_2-exchange, the rights $\Delta^h\left(k^h(p_2), p_2\right)$ with the indicated savings $k^h(p_2)$ determine the participation fee (paid or received at $t = 1$). Saving $k^h(p_2)$ and consumption $\mathbf{c}^h_1(p_2)$ are then also purchased by type h in the $t = 1$ spot market at given spot market prices. We then move to the corresponding p_2 spot market at $t = 2$. The rights $\Delta^h\left(k^h(p_2), p_2\right)$ will be equal to what agent h will want to do at $t = 2$ since the rights are excess demands by construction, i.e., $\Delta^h\left(k^h(p_2), p_2\right) \equiv \tau^{h*}_{w2}\left(k^h(p_2), p_2\right)$. So the solution is time consistent.[19]

Definition 6. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h(p_2), \delta^h(p_2), \boldsymbol{\tau}^h(p_2)\right]$, price of good z at $t = 1$, p_1, spot prices p_2 for active and potential spot markets at $t = 2$, and the prices of the rights to trade $\left[\mathbf{P}_\Delta(p_2)\right]_{p_2}$ such that

(i) at $t = 2$, taking $k^h(p_2)$ as predetermined and spot price p_2 as given, agent type $h = a, b$ solves for trades $\boldsymbol{\tau}^h(p_2)$ subject to budget (10) to define value function $V^h\left(k^h(p_2), p_2\right)$;

(ii) at $t = 1$, agent type h takes spot prices p_1 and the price of rights $P_\Delta(p_2)$ as given and solves for the choice of p_2 and associated $\mathbf{x}^h(p_2)$ to

$$\max_{\left[\mathbf{x}^h(p_2), \delta^h(p_2)\right]_{p_2}} \sum_{p_2} \delta^h(p_2)\left[u^h\left(c^h_{w1}(p_2), c^h_{z1}(p_2)\right) + V^h\left(k^h(p_2), p_2\right)\right] \tag{27}$$

[19]Equivalently, we could require that all trade in each of the exchanges be done with a central counter party, CCP, who becomes the buyer for every seller and the seller for every buyer. The CCP as a broker-dealer has to make sure that all trades clear and that saving and consumptions at $t = 1$ are funded. The continuum agent assumption removes any uncertainty. This equivalent formulation is useful when we have multiple active exchanges with the mixtures, as presented in a numerical example in Kilenthong and Townsend (2014).

subject to the non-negative saving constraint for the agent type h indexed by p_2, (21) and the budget constraint in the first period $t = 1$

$$\sum_{p_2} \delta^h(p_2)\left[c^h_{w1}(p_2) + p_1 c^h_{z1}(p_2) + p_1 k^h(p_2) + P_\Delta(p_2)\Delta^h\left(k^h(p_2), p_2\right)\right] \leq e^h_{w1} + p_1 e^h_{z1}, \tag{28}$$

Note that the agent does not choose the trades of the others nor takes them as given and, unlike the incomplete markets, the budget now includes prices of rights $P_\Delta(p_2)$ and demand for rights $\Delta^h\left(k^h(p_2), p_2\right)$;[20]

(iii) market-clearing conditions hold: for good w, (29), for good z (30), for rights, (31), and for trades, (32),

$$\sum_{p_2}\sum_h \delta^h(p_2)\alpha^h c^h_{w1}(p_2) = \sum_h \alpha^h e^h_{w1}, \tag{29}$$

$$\sum_{p_2}\sum_h \delta^h(p_2)\alpha^h\left[c^h_{z1}(p_2) + k^h(p_2)\right] = \sum_h \alpha^h e^h_{z1}, \tag{30}$$

$$\sum_h \alpha^h \delta^h(p_2)\Delta^h\left(k^h(p_2), p_2\right) = 0, \forall p_2, \tag{31}$$

$$\sum_h \delta^h(p_2)\alpha^h \tau^h_{\ell 2}(p_2) = 0, \forall \ell = w, z; p_2. \tag{32}$$

Using steps similar to those in the preceding section for the market maker, we can then write the necessary conditions for type h maximization as follows.

$$p_1 = \frac{u^h_{z1}}{u^h_{w1}} = \frac{u^h_{z2}}{u^h_{w1}}R + \frac{\eta^h(p_2)}{u^h_{w1}} - P_\Delta(p_2)\Delta^h_k\left(k^h(p_2), p_2\right), \forall h = a, b. \tag{33}$$

which is starkly similar to (25) of the planner as market maker. The last term in equations (25) and (33) is the externality correction term that comes from the decentralizing way which incorporates the rights to trade. Indeed, the two equations, (25) and (33), are identical when we match the Lagrange multipliers and prices from the planner problem and the new decentralized equilibrium in Definition 6 using the following conditions: $P_\Delta(p_2) = \frac{\mu_\Delta(p_2)}{\mu_{w1}}$ and $\eta^h = \frac{\mu^h}{\lambda^h \alpha^h}$. Note that the Pareto weights λ^h associated with the competitive equilibrium can be recovered using the following condition: $\lambda^h = \frac{\mu_{w1}}{\eta^h_{bc,1}}$, where μ_{w1} is the Lagrange multiplier on (22), and $\eta^h_{bc,1}$ is the Lagrange multiplier on budget constraints (28). The competitive

[20]When $0 < \delta^h(p_2) < 1$, the objective function (27) is expected utility and the LHS of (28) is expected expenditure. However, prices will reflect actuarial values and so (28) is deterministic and holds as exact.

equilibrium with rights picks out one of the Pareto optimal allocations as a solution to the market maker problem, namely at Pareto weights λ^h which do not require lump-sum taxes and transfers.

Similar to the market maker problem, as in equation (26), the choice of p_2, for each h with fraction $\delta^h(p_2) > 0$, satisfies the following condition[21] (see Online Appendix B for the derivation):

$$\left(\frac{1}{\eta^h_{bc,1}}\right) V^h_p\left(k^h(p_2), p_2\right) = P_\Delta(p_2)\, \Delta^h_p\left(k^h(p_2), p_2\right) \tag{34}$$

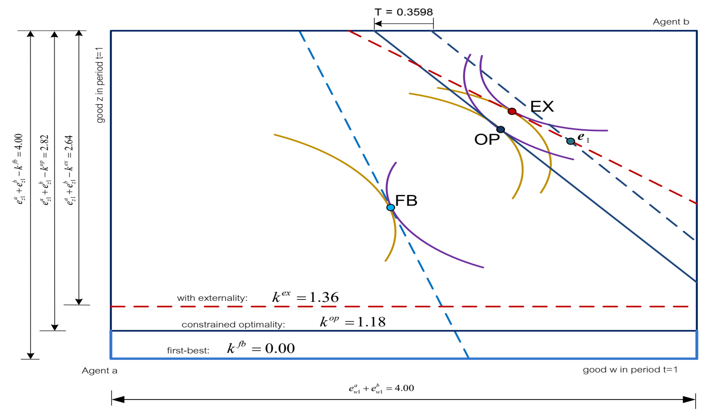
For the numerical example, the competitive equilibrium with rights to trade has *one and only one active exchange*, $p_2^{op} = 0.5974$, even though all exchanges are available in principle a priori for trade. That is, in equilibrium, both types optimally choose the same p_2-exchange at $t = 1$ and hence the same p_2 spot market with, $p_2^{op} = 0.5974$. Table 4 presents equilibrium prices/fees of rights to trade, that is $P_\Delta(p_2)$ not only for p_2^{op} but also other, different spot price levels p_2.[22] Note again that the prices/fees of non-active spot markets are available, but facing such prices, agents do not want to trade in them.[23] Again, both types choose $p_2^{op} = 0.5974$. Agent as origin moves as a function of saving; for example, with the pecuniary externality, more is saved, so points in the box reflect less consumption of good z. The externality (EX) point is achieved by movement along a budget line at equilibrium prices through the endowment. The slope is determined by the ratio of good z to good w and hence is relatively flat, with less of good z available for consumption. The constrained-optimal (OP) line has a steeper slope and shifts relative to EX in favor of agent b as compensation for the type selling rights. The first-best (FB) point has no saving and an equal amount of consumption for both types.

An agent type a comes into the spot market at $t = 2$ with good z in storage. So, type a buys the right to buy good w in amount $\Delta^a(p_2^{op}) = 0.2970$ (in exchange for good z, which

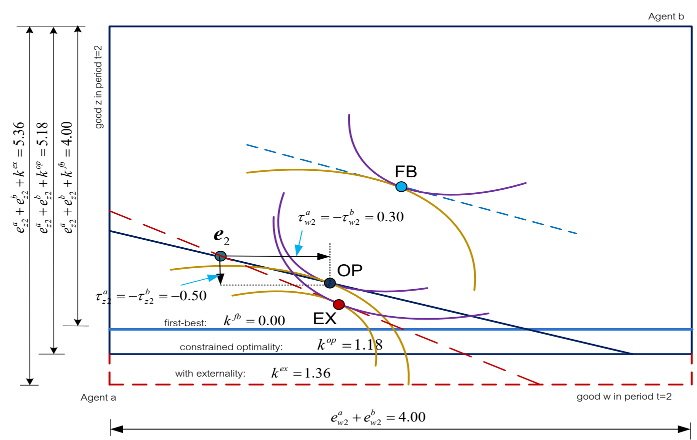
[21]If this condition were at an inequality, it would determine the direction of adjustment.

[22]The shadow prices for inactive markets are derived sequentially, as noted in the discussion below (19): solve sub-problems locally, for a given p_2, and solve the global problem of choosing p_2. This has deeper roots in Lagrangian mechanics.

[23]Prices used are based on shadow prices from the planner problem, hence true marginal costs. As in standard price theory, markets in some goods can be cleared at prices implying zero activity, as when prices at marginal costs are strictly larger than the willingness to pay. These equilibrium prices can be indeterminate in a certain range in the sense that marginal cost prices can be lowered a bit but not impact the allocation.



(a) Edgeworth box for $t = 1$.



(b) Edgeworth box for $t = 2$.

Figure 1: Figure 1(a) displays the allocations of goods w and z at $t = 1$. Agent a's origin moves as a function of saving; for example, with the pecuniary externality, more is saved, so points in the box reflect less consumption of good z. The externality (EX) point is achieved by movement along a budget line at equilibrium prices through the endowment. The slope is determined by the ratio of good z to good w and hence is relatively flat, with less of good z available for consumption. The constrained-optimal (OP) line has a steeper slope and shifts relative to EX in favor of agent b as compensation for the type selling rights. The first-best (FB) point has no saving and an equal amount of consumption for both types. Similarly, Figure 1(b) displays the allocations of goods w and z at $t = 2$. The slope of the OP line is flatter here because there is less saving carried to period $t = 2$.

Table 4: Equilibrium prices of rights to trade in spot markets $P_{\Delta}(p_2)$ at price p_2.

	$p_2 = 0.5770$	$p_2^{op} = 0.5974$	$p_2 = 0.6181$
$P_{\Delta}(p_2)$	1.1383	1.2116	1.2840

is sold of course), where $\Delta^h(p_2^{op}) \equiv \Delta^h\left(k^h(p_2^{op}), p_2^{op}\right)$. Here "op" designates constrained-optimal. This makes sense as agent type a is doing the saving in good z and there is too much saving in the competitive equilibrium with the externality. As with pollution, this is how the externality is corrected. On the other hand, an agent type b will be paid for her willingness to choose that market $p_2^{op} = 0.5974$. Agent type b is facing a higher price of the good z, and good z will be purchased so excess demand for good w is negative. But there is compensation. In particular, with $\Delta^b(p_2^{op}) = -0.2970$ and $P_{\Delta}(p_2^{op}) = 1.2116$, agent b is receiving $-P_{\Delta}(p_2^{op})\Delta^b(p_2^{op}) = 0.3598$ in period $t = 1$ for being in the spot market $p_2^{op} = 0.5974$. Graphically, this shifts her budget line outward at $t = 1$ by $T = 0.3598$, hence in the direction of being less constrained. This is displayed in Figure 1 along with the allocations in both dates for the competitive equilibrium with externalities (EX), the constrained optimal allocation (OP), and the first best (FB).[24]

3 General Economy

This section presents an extension of the leading example by adding uncertainty, and traded securities, yet allowing for market incompleteness and collateral constraints.[25] Consider an economy with S possible states of nature at $t = 2$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. Each agent type h is endowed with $\left(e_{w1}^h, e_{z1}^h\right)$ at date $t = 1$ and $\left(e_{w2s}^h, e_{z2s}^h\right)$ in state s at date $t = 2$. The utility functions u^h are strictly concave with other regularity conditions. There are J securities available for purchase or sale at $t = 1$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets at $t = 2$ where $D_{js} \in \mathbb{R}_+$ is the payoff

[24] Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium relative to the markets without rights. Thus if nothing else were done, internalizing the externality would be beneficial to an agent type b (constrained agent) but harmful for an agent type a. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we state in Section 3.1 and prove in Appendix 4.

[25] We provide numerical examples of an economy with active security holdings and an economy with incomplete markets in Kilenthong and Townsend (2014).

of asset j in units of good w (the numeraire good) in state $s = 1, 2, \ldots, S$. Here we do not include securities paying in good z as there is trade in the two goods in spot markets at price p_{2s} in terms of the numeraire so these are not needed. Let θ_j^h denote the amount of the j^{th} security acquired by an agent of type h at $t = 1$ with $\boldsymbol{\theta}^h \equiv \left[\theta_j^h\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise to deliver at $t = 2$. The collateral constraint in state s at date $t = 2$ states that there must be sufficient collateral in value to honor all promises:

$$p_{2s} R_s k^h + \sum_j D_{js} \theta_j^h \geq 0, \forall s, \tag{35}$$

where R_s is the state contingent return on the collateral. Equation (35) can be rewritten with securities $\theta_j^h > 0$ as investments with payouts D_{js} added to the value of collateral in terms of the numeraire on the left hand side and the securities $\theta_j^h < 0$ as promises with obligations D_{js} on the right hand side. This is a generalized version of $k^h \geq 0$ in the saving economy. More general obstacle-to-trade constraints applicable to our market-based approach are presented in the Online Appendix H.

With potentially incomplete security markets, a given security traded at $t = 1$ has implications in general for most if not all spot prices at $t = 2$. This is one source of externalities. In addition, promises are at least potentially backed by collateral good z, which is carried over to $t = 2$, another source of externalities as in the saving example. To internalize these externalities, we thus need rights to trade indexed by the vector of spot prices $\mathbf{p} = [p_{2s}]_s$ over all states s at date $t = 2$.[26] That is, what we now term $\mathbf{p}$-exchanges must naturally deal with S spot markets as a bundle. As a result, all objects are indexed by entire vector $\mathbf{p}$. This is where there is a subtle difference from the saving economy. Note also that our solution is not about completing markets as illustrated in the numerical example in Online Appendix G.

Let $Q_j(\mathbf{p})$ denote the price of security j at $t = 1$ executed at $t = 2$ in an exchange $\mathbf{p}$ with vector $\mathbf{Q}(\mathbf{p}) \equiv [Q_j(\mathbf{p})]_j$. The rights to trade acquired an exchange $\mathbf{p}$, to be executed at $t = 2$ at the designated price, is denoted by a vector of rights $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta_s^h(\mathbf{p})\right]_s$, where for the component at state s,

$$\Delta_s^h(\mathbf{p}) \equiv \tau_{w2s}^{h*}\left(p_s, k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p})\right), \forall s, \tag{36}$$

[26]Henceforth, bold typeface refers to a vector.

which is the standard excess demand for good w in the spot market s at $t = 2$ for an agent type h holding collateral $k^h(\mathbf{p})$, securities $\boldsymbol{\theta}^h(\mathbf{p})$, and being in an exchange $\mathbf{p}$. For brevity, we write this right $\Delta_s^h(\mathbf{p})$ as a function of the spot prices $\mathbf{p}$ only on the left hand side of (36) even though the excess demand depends on the pre-trade position coming from collateral/savings and securities. Let $\mathbf{x}^h = \left[\mathbf{c}_1^h(\mathbf{p}), k^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{\mathbf{p}}$ denote a typical bundle or allocation for an agent type h, with $t = 1$ consumption, saving, and security holdings, the choice $\mathbf{p}$-exchange, and rights consistent with excess demands all chosen by the planner.

As illustrated in the example and formally proved in the following section, a constrained optimal allocation, a solution to the Pareto program, can be decentralized in a competitive equilibrium with rights to trade, a generalized version of the one defined in Section 2.5.

Definition 7. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h, \boldsymbol{\tau}^h, \delta^h\right]_h$, price of good z at $t = 1$, p_1, spot prices $\mathbf{p} = [p_{2s}]_s$ for active and potential spot markets at $t = 2$, and the prices of securities and the rights to trade $[\mathbf{Q}(\mathbf{p}), \mathbf{P}_\Delta(\mathbf{p})]_{\mathbf{p}}$ such that (i) in state s at date $t = 2$, taking $\left(k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{p}\right)$ as given, agent type $h = a, b$ solves

$$V_s^h\left(k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{p}\right) = \max_{\tau_{w2s}^h(\mathbf{p}), \tau_{z2s}^h(\mathbf{p})} u^h\big(e_{w2s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{w2s}^h(\mathbf{p}), e_{z2s}^h + R_s k^h(\mathbf{p}) + \tau_{z2s}^h(\mathbf{p})\big) \tag{37}$$

subject to the budget constraint in period $t = 2$,

$$\tau_{w2}^h(\mathbf{p}) + p_{2s}\tau_{z2}^h(\mathbf{p}) = 0; \tag{38}$$

(i) at date $t = 1$, for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p}), \delta^h(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\mathbf{x}^h, \delta^h} \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u^h\big(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})\big) + \sum_s \pi_s V_s^h\left(k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{p}\right)\right] \tag{39}$$

subject to the budget constraints in the first period, which now includes securities $\boldsymbol{\theta}^h$ at vector of prices $\mathbf{Q}$ and vector of rights $\boldsymbol{\Delta}^h$ at vector of prices $\mathbf{P}_\Delta$

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{w1}^h(\mathbf{p}) + p_1\left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p})\right] + \mathbf{Q}(\mathbf{p}) \cdot \boldsymbol{\theta}^h(\mathbf{p}) + \mathbf{P}_\Delta(\mathbf{p}) \cdot \boldsymbol{\Delta}^h(\mathbf{p})\right] \leq e_{w1}^h + p_1 e_{z1}^h, \tag{40}$$

non-negative saving constraints and collateral constraints, respectively,

$$\delta^h(\mathbf{p})\, k^h(\mathbf{p}) \geq 0, \forall h; \mathbf{p}, \tag{41}$$

$$\delta^h(\mathbf{p}) \left[p_{2s}R_s k^h(\mathbf{p}) + \sum_j D_{js}\theta_j^h(\mathbf{p})\right] \geq 0, \forall h; \mathbf{p}; s, \tag{42}$$

(ii) market-clearing conditions for good w, (43); good z (44); securities θ_j^h, (45), trades $\tau_{\ell 2}^h$, (47), and market-clearing conditions for $\Delta_s^h(\mathbf{p})$, (46), for all $\mathbf{p}$, respectively,

$$\sum_{\mathbf{p}}\sum_{h}\delta^h(\mathbf{p})\,\alpha^h c_{w1}^h(\mathbf{p}) = \sum_h \alpha^h e_{w1}^h; \tag{43}$$

$$\sum_{p_2}\sum_{h}\delta^h(\mathbf{p})\,\alpha^h\left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p})\right] = \sum_h \alpha^h e_{z1}^h; \tag{44}$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h\theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}; \tag{45}$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h\Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \tag{46}$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h\tau_{\ell 2}^h(\mathbf{p}) = 0, \forall \ell = w, z; \mathbf{p}; \tag{47}$$

hold.

The Pareto program with Pareto weights $\left[\lambda^h\right]_h$, analogous to the market maker problem (20), is defined as follows.

$$\max_{\left[\mathbf{x}^h,\delta^h(\mathbf{p})\right]_h}\sum_{h,\mathbf{p}}\lambda^h\alpha^h\delta^h(\mathbf{p})\left[u^h\left(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})\right) + \sum_s \pi_s V_s^h\left(k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{p}\right)\right]$$

subject to non-negative saving constraints (41); collateral constraints (42), the resource constraints for good w (43) and good z (44) in period $t = 1$, the adding-up constraints for securities (45), and the consistency constraints (46).

3.1 Welfare Theorems and Existence Theorem

By a suitable extension of the commodity space that allows mixture representations as formalized in the appendix, the economy becomes a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption sets are compact and convex, and the utility functions are linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists.

For the first welfare theorem, the standard proof-by-contradiction argument is used. See the Online Appendix C for the proof. We also assume that there is a nonsatiation point in the consumption set. Based on this non-satiation assumption, we have the formal statement:

Theorem 1. *With non-satiation of preferences, a competitive equilibrium with rights to trade in* $\mathbf{p}$*-exchanges is constrained Pareto optimal.*

The second welfare theorem can be established by matching first-order conditions of individual's and planner's problems. Though this theorem deals with any constrained Pareto optimal allocation, one of them corresponds to the competitive equilibrium without transfers. The standard proof applies. Any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers (see the Appendix). The formal statement:

Theorem 2. *Any constrained Pareto optimal allocation with strictly positive Pareto weights* $\lambda^h > 0, \forall h$ *can be supported as a competitive equilibrium with rights to trade with transfers.*

Finally we have the existence theorem. We use Negishi's mapping method (Negishi, 1960). The proof benefits from the second welfare theorem, that the solution to the Pareto program is a competitive equilibrium with transfers. We then show that a fixed-point of the mapping exists and represents a competitive equilibrium without transfers and, using the mapping, is constrained optimal. See the Online Appendix C for the proof. The formal statement:

Theorem 3. *With local non-satiation of preferences and positive endowments, a competitive equilibrium with rights to trade exists.*

In addition, we can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when markets for rights to trade are introduced. See more details in the Online Appendix F.

The goal of the paper was to examine if the solution to the planner problem as a constrained-optimal target allocation can be achieved in competitive markets, in particular whether we can do this for pecuniary externalities. We answered affirmatively. In the context of the featured example savings economy, one can work backwards, take our market-determined solution, and then interpret it as a tax on saving, along with lump-sum taxes and redistributions based on ownership of endowments, as in the Online Appendix E. However, this can be misleading; one might draw the wrong lessons. First, if one gives the planner the power to redistribute wealth arbitrarily, one can violate the no-borrowing constraint and hence violate a key constraint on the planner problem. Second, with non-homothetic utility and incomplete securities which do not span the space of returns, tax schedules with rebates can have high dimensionality, as a function of the number of securities, and are complicated, as prices are not monotonic and move in subtle ways with type specific pre-trade positions. In contrast, our market-based solution with rights deals directly in the space of prices, much in the spirit of a reduced form or sufficient statistic argument.

4 Conclusion

Our solution concept extends to many other well-known environments in the literature that have prices in constraints beyond the role of prices in budget constraints. The collateral constraints are featured in the general model, but more generally there are sets of obstacle-to-trade constraints which include as arguments not only consumption, securities, spot trades, and inputs and outputs from production, but also vectors of prices. In the Online Appendix H, we write out these constraints for additional prototype economies mentioned in the introduction.

It is natural to ask how the markets we have described would come about and how prices would be determined. Our answer is two-fold.

First, the features of institutions that our model requires are already out there and in use, in other contexts. Securities are held, maintained, and registered on electronic book entry systems and direct transfers of securities are made through specified utilities. Further, it is not uncommon that only some agents are allowed to participate, so, the necessary exclusivity required by the theory is not hard to imagine.

Secondly, our methods for proofs of the existence of Walrasian competitive equilibrium and the welfare theorems consist of converting the underlying economy with collateral, spot and forward markets, and rights to the notation of the standard Arrow, Debreu, McKenzie general equilibrium model. Thus, with that notation as a starting point, one can as in Townsend (1983) have broker dealers as intermediaries, market makers who call out prices for the commodity points and compete for the right to engage in exclusive trade with clients, buying and selling, potentially taking net positions themselves. With a continuum of traders this results in the competitive equilibrium.[27]

[27]There is a related literature on the implementation of Walrasian equilibria as the outcome of market making games as in Dubey (1982), and a mechanism design literature (e.g., Allen and Jordan, 1998) regarding a minimum communication system to implement the Walrasian allocation which suggests that prices are not enough. This is also related to a computer science literature for computational algorithms that achieve the Walrasian equilibria, as in Echenique and Wierman (2011), consulting the excess demand oracle a judiciously

Potential difficulties that will have to be thought through include incomplete enumeration of future states, in which case we hope our solution works as an approximation. Related, Jeremy Stein has written about ex ante fees, a price based mechanism, for the use of a central bank credit liquidity facility, which might be contingent on some adverse states such as financial shocks, when liquidity is at a premium. Another difficulty would be vested interests that resist market reform without compensation, though this issue is not new nor peculiar to the situation here. Finally, there could be a problem with inactive exchanges. The theory requires that traders can choose any **p**-exchange they want, and we do not want the inactive ones to be eliminated prematurely. Ex ante we do not know which exchanges these will be. As Stein noted, the use of rights priced with fees in financial markets can help deal with situations in which even well informed regulators cannot know the exact requirements that might be needed.[28]

References

Acemoglu, D. and A. Simsek (2012). Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading. *Working Paper*.

Allen, B. E. and J. S. Jordan (1998). The existence of rational expectations equilibrium: A retrospective. In *Organizations with Incomplete Information: Essays in Economic Analysis, A Tribute to Roy Radner*, pp. 42–60. Cambridge University Press.

Arrow, K. (1969). The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Nonmarket Allocation. *The Analysis and Evaluation of Public Expenditure: The PPB System 1*, 59–73.

Bisin, A. and P. Gottardi (2006). Efficient Competitive Equilibria with Adverse Selection. *Journal of Political Economy 114*(3), 485–516.

Coase, R. H. (1960). The Problem of Social Cost. *Journal of Law and Economics 3*, 1.

Debreu, G. (1954). Valuation Equilibrium and Pareto Optimum. *Proceedings of the National Academy of Sciences of the United States of America 40*(7), 588.

limited number of time.

[28]See the article at https://www.federalreserve.gov/newsevents/speech/stein20130419a.htm.

Diamond, D. and P. Dybvig (1983). Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy 91*(3), 401.

Dubey, P. (1982). Price-Quantity Strategic Market Games. *Econometrica*, 111–126.

Echenique, F. and A. Wierman (2011). Finding a Walrasian Equilibrium is Easy for a Fixed Number of Agents. *Working Paper*.

Geanakoplos, J. and H. Polemarchakis (1986). Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete. *Uncertainty, Information and Communication: Essays in Honor of KJ Arrow 3*, 65–96.

Gorton, G. and A. Metrick (2012). Securitized Banking and the Run on Repo. *Journal of Financial Economics 104*(3), 425–451.

Greenwald, B. and J. Stiglitz (1986). Externalities in Economies with Imperfect Information and Incomplete Markets. *Quarterly Journal of Economics 101*(2), 229–264.

Hart, O. D. and L. Zingales (2013). Inefficient Provision of Liquidity. Technical report, Working Paper.

Jacklin, C. (1987). Demand Deposits, Trading Restrictions, and Risk Sharing. *Contractual Arrangements for Intertemporal Trade*, 26–47.

Kilenthong, W. T. and R. M. Townsend (2011). Information-Constrained Optima with Retrading: An Externality and its Market-Based Solution. *Journal of Economic Theory 146*(3), 1042–1077.

Kilenthong, W. T. and R. M. Townsend (2014). Segregated Security Exchanges with Ex Ante Rights to Trade. Technical report, National Bureau of Economic Research.

Lindahl, E. (1958). Just Taxation–A Positive Solution. In R. A. Musgrave and A. T. Peacock (Eds.), *Classics in the theory of public finance*. Macmillan, London.

Lorenzoni, G. (2008). Inefficient Credit Booms. *Review of Economic Studies 75*(3), 809–833.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995). *Microeconomic Theory*, Volume 1. Oxford university press New York.

Meade, J. E. (1952). External Economies and Diseconomies in a Competitive Situation. *The Economic Journal*, 54–67.

Negishi, T. (1960). Welfare Economics and Existence of an Equilibrium for a Competitive Economy. *Metroeconomica 12*(2-3), 92–97.

Prescott, E. C. and R. M. Townsend (1984a). General Competitive Analysis in an Economy with Private Information. *International Economic Review 25*(1), 1–20.

Prescott, E. C. and R. M. Townsend (1984b). Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard. *Econometrica 52*(1), 21–45.

Townsend, R. M. (1983). Theories of intermediated structures. *Carnegie-Rochester Conference Series on Public Policy 18*, 221–272.

Appendix: Proofs

As noted in the text, when the planner is choosing prices we can lose concavity of the programs and it can be hard to show equivalence of planner problem and decentralized competitive equilibria. However, as already anticipated in the $\delta^h(p_2)$ notation interpreted as lotteries, this is easily fixed with the introduction of lotteries over all the entire underlying commodity points both in the planner problem and for the agents.[29] Thus, as in Prescott and Townsend (1984b), let $x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0$ denote the probability of receiving period $t = 1$ consumption $\mathbf{c}_1$, collateral k, securities $\boldsymbol{\theta}$, period $t = 2$ spot trades $\boldsymbol{\tau}$, and being in exchanges indexed by $\mathbf{p} \equiv [p_{2s}]_s$ with rights to trade $\boldsymbol{\Delta}$. We write again the spot market budget, the non-negative saving constraint and the collateral constraints in state s at date $t = 2$:

$$\tau_{w2s} + p_{2s}\tau_{z2s} = 0, \forall s;\ k \geq 0;\ p_{2s}R_s k + \sum_j D_{js}\theta_j \geq 0, \forall s. \tag{48}$$

[29]Lotteries are helpful to convexify the problems even when the ultimate outcome is a degenerate lottery, deterministic choices, as in the notation of the text. It can happen is some environments that the lotteries are nontrivial, assigning various fractions of a given type to distinct price markets. See section 4.2 in Kilenthong and Townsend (2014) for an example.

Accordingly, we impose the following condition on a probability measure:

$$x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0 \text{ if } (\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \text{ satisfies (36), (48),} \tag{49}$$

and zero otherwise. The consumption possibility set of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}^n_+ : \sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}} x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) = 1, \text{ and (49) holds} \right\}. \tag{50}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option. For notational purposes, let $\mathbf{w} \equiv (\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be a typical bundle, and the utility derived from it for an agent type h is defined by $U^h(\mathbf{w}) = u^h\big(c_{w1}, c_{z1}\big) + \sum_s \pi_s u^h\big(e^h_{w2s} + \sum_j D_{js}\theta^h_j + \tau_{w2s}, e^h_{z2s} + R_s k + \tau_{z2s}\big)$. Then, we have the maximization problem for agents as part of the definition of equilibrium: for each h, $\mathbf{x}^h \in X^h$ solves

$$\max_{\mathbf{x}^h \in X^h} \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w}) \tag{51}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 1$ budget constraint, that the valuation of endowments sold provides revenue for purchase of the lotteries.

$$\sum_{\mathbf{w}} P(\mathbf{w})\, x^h(\mathbf{w}) \quad \leq \quad e^h_{w1} + p_1 e^h_{z1}, \tag{52}$$

taking price of good z at $t = 1$, p_1, and prices of lottery, $P(\mathbf{w})$ as given.

We introduce broker dealers that run the $\mathbf{p}$-exchanges and deal with households for trades in securities, collateral, rights to trade and spot trades. The consumption $\mathbf{c}_1$ and collateral k commitments are sold but must be funded by the requisite amount of consumption goods and collateral. Securities, rights and spot trades do not require resources but are cleared by the broker-dealers. There are constant returns to scale in these activities so it is as if there were one representative broker-dealer. Let $b(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ denote the quantity of commitment to provide $(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. See Prescott and Townsend (1984a) for the introduction of broker-dealer. The broker-dealer takes prices p_1 and $P(\mathbf{w})$ as given and supplies $\mathbf{b}$ to solve the following profit maximization problem:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} b(\mathbf{w})\left[P(\mathbf{w}) - c_{w1} - p_1 c_{z1} - p_1 k\right] \tag{53}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} b\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\theta_j = 0,\ \forall j;\mathbf{p}, \tag{54}$$

$$\sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} b\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\tau_{\ell 2s} = 0,\ \forall s;\ell;\mathbf{p}, \tag{55}$$

$$\sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} b\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\Delta_s = 0\ ,\forall s;\mathbf{p}. \tag{56}$$

Market clearing conditions in the two consumption goods is standard, purchased consumptions and collateral by the broker-dealer equals supply of endowments from the households:

$$\sum_{\mathbf{w}} b\left(\mathbf{w}\right)c_{w1} = \sum_{h}\alpha^h e^h_{w1}, \tag{57}$$

$$\sum_{\mathbf{w}} b\left(\mathbf{w}\right)\left[c_{z1}+k\right] = \sum_{h}\alpha^h e^h_{z1}. \tag{58}$$

The net demand for contracts by households, allowing non-degenerate fractions in the population, equals the supply of contracts by the broker-dealer:

$$\sum_{h}\alpha^h x^h\left(\mathbf{w}\right) = b\left(\mathbf{w}\right),\ \forall\mathbf{w}. \tag{59}$$

See Kilenthong and Townsend (2014) for a particular clarified example of what broker-dealers in the context of an environment with multiple active exchanges.

Definition 8. A competitive equilibrium with rights to trade (with mixtures) is a specification of allocation $\left(\mathbf{x}^h,\mathbf{b}\right)$, and prices $\left(p_1,P\left(\mathbf{w}\right)\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves the utility maximization problem (51) taking prices as given;

(ii) for the broker-dealer, $\mathbf{b}$ solves the maximization problem (53), taking prices as given;

(iii) market clearing conditions (57)-(59) hold.

The Pareto problem with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h\in X^h\right]_h}\sum_{h}\lambda^h\alpha^h\sum_{\mathbf{w}}x^h\left(\mathbf{w}\right)U^h\left(\mathbf{w}\right) \tag{60}$$

subject to

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, c_{w1} = \sum_h \alpha^h e_{w1}^h, \tag{61}$$

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, [c_{z1} + k] = \sum_h \alpha^h e_{z1}^h, \tag{62}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \tau_{\ell 2s} = 0, \forall \ell; s; \mathbf{p}, \tag{63}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \theta_j = 0, \forall j; \mathbf{p}, \tag{64}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \Delta_s = 0, \forall s; \mathbf{p}. \tag{65}$$

Proof of The Second Welfare Theorem (Theorem 2)

Proof of Theorem 2. Since the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof is divided into three steps.

(i) Kuhn-Tucker conditions for a compensated equilibrium allocation: Let $\hat{\gamma}_U^h$ and $\hat{\gamma}_l^h$ be the Lagrange multiplier for the reservation-utility constraint, and for the probability constraint, respectively. The optimal condition for $x^h(\mathbf{w})$ is given by

$$\hat{\gamma}_U^h U^h(\mathbf{w}) \leq P(\mathbf{w}) + \hat{\gamma}_l^h, \tag{66}$$

where the inequality holds with equality if $x^h(\mathbf{w}) > 0$. The optimal condition for the broker-dealer's profit maximization problem implies that, for any typical bundle $\mathbf{w}$,

$$P(\mathbf{w}) \leq c_{w1} + p_1[c_{z1} + k] + \sum_j \widehat{Q}_j(\mathbf{p})\,\theta_j + \sum_s \sum_\ell \hat{p}_\ell(\mathbf{p}, s)\,\tau_{\ell 2s} + \sum_s \widehat{P}_\Delta(\mathbf{p}, s)\,\Delta_s,$$

where $\widehat{Q}_j(\mathbf{p})$, $\hat{p}_\ell(\mathbf{p}, s)$ and $\widehat{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (54)-(56). The condition holds with equality if $b(\mathbf{w}) > 0$.

(ii) Kuhn-Tucker conditions for Pareto optimal allocations: A solution to the Pareto program satisfies the following optimal condition

$$\lambda^h U^h(\mathbf{w}) \leq \tilde{p}_{w1} c_{w1} + \tilde{p}_{z1}[c_{z1} + k] + \sum_j \widetilde{Q}_j(\mathbf{p})\,\theta_j + \sum_s \sum_\ell \tilde{p}_\ell(\mathbf{p}, s)\,\tau_{\ell 2s} + \sum_s \widetilde{P}_\Delta(\mathbf{p}, s)\,\Delta_s + \tilde{\gamma}_l^h,$$

where $\tilde{\gamma}_l^h$ is the Lagrange multiplier for the probability constraint, and $\tilde{p}_{w1}$, $\tilde{p}_{z1}$, $\widetilde{Q}_j(\mathbf{p})$, $\tilde{p}_\ell(\mathbf{p}, s)$ and $\widetilde{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (61)-(65), respectively. Again, the condition holds with equality if $x^h(\mathbf{w}) > 0$.

(iii) Matching dual variables and prices: We can now set $\hat{\gamma}_U^h = \frac{\lambda^h}{\tilde{p}_{w1}}$, $p_1 = \frac{\tilde{p}_{z1}}{\tilde{p}_{w1}}$, $\widehat{Q}_j(\mathbf{p}) = \frac{\widetilde{Q}_j(\mathbf{p})}{\tilde{p}_{w1}}$, $\hat{p}_\ell(\mathbf{p}, s) = \frac{\tilde{p}_\ell(\mathbf{p},s)}{\tilde{p}_{w1}}$ and $\widehat{P}_\Delta(\mathbf{p}, s) = \frac{\widetilde{P}_\Delta(\mathbf{p},s)}{\tilde{p}_{w1}}$, and $\hat{\gamma}_l^h = \frac{\tilde{\gamma}_l^h}{\tilde{p}_{w1}}$. These matching conditions imply that the optimal conditions of the Pareto program are equivalent to the optimal conditions for consumers' and broker-dealer's problems in the compensated equilibrium. To sum up, any Pareto optimal allocation is a compensated equilibrium.

We can show that any compensated equilibrium, corresponding to $\lambda^h > 0$, is a competitive equilibrium with transfers using the cheaper point argument, which is obvious given the strictly positive Pareto weight and strictly positive endowment. Using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. $\square$